NNT : 2015SACLS211

UNIVERSITÉ PARIS-SACLAY

École doctorale de mathématiques Hadamard (EDMH, ED 574)

Établissement d'inscription : Université Paris-Sud

Laboratoire d'accueil : Laboratoire de mathématiques d'Orsay, UMR 8628 CNRS

THÈSE DE DOCTORAT

Spécialité : Mathématiques fondamentales

Céline Abraham

Cartes aléatoires et serpent brownien

Date de soutenance : 11 décembre 2015

Jury de soutenance :	NICOLAS CURIEN	(Université Paris-Sud)	Examinateur
	JEAN-FRANÇOIS DELMAS	(ENPC)	Rapporteur
	BÉNÉDICTE HAAS	(Université Paris 13)	Présidente
	JEAN-FRANÇOIS LE GALL	(Université Paris-Sud)	Directeur de thèse
	GRÉGORY MIERMONT	(ÉNS de Lyon)	Rapporteur

Table des matières

Chapitre 1

Introduction

Contenu de ce chapitre

Dans cette introduction, nous décrivons les principaux résultats de cette thèse en les replaçant dans l'histoire du domaine. Les résultats que nous avons obtenus apparaissent encadrés et notés avec les lettres capitales A,B,C, etc.

1.1 Cartes planaires

1.1.1 Définition des cartes

On commence par définir les cartes planaires. Un multigraphe G est un couple (V, E) où V est un ensemble de sommets et E est un ensemble d'arêtes. Si les deux extrémités d'une arête coïncident, on dit que c'est une boucle. Si plusieurs arêtes distinctes ont les deux mêmes extrémités, on parle d'arête multiple. Dans la suite on dira graphe au lieu de multigraphe.

Definition 1.1.1. *Une carte planaire finie m est une classe d'équivalence d'un graphe fini connexe (on autorise les boucles et les arêtes multiples) plongé dans la sphère, modulo les homéomorphismes de la sphère qui préservent l'orientation.*

Les faces d'une carte planaire correspondent aux composantes connexes du complémentaire de l'union des arêtes. On note F l'ensemble des faces.

Dans le cadre de notre étude, on considère toujours des cartes planaires. Un des premiers résultats importants sur les cartes est la formule d'Euler, qui relie le nombre de sommets, d'arêtes et de faces d'une carte. Pour une carte planaire, on a

$$\mathrm{Card}(V) + \mathrm{Card}(F) - \mathrm{Card}(E) = 2. \tag{1.1}$$

On introduit le degré d'une face de la carte. Le degré d'une face est défini comme le nombre de demi-arêtes qui bordent cette face. Cela signifie que si une arête est contenue entièrement dans une face, elle compte double dans le calcul du degré.

Pour un entier $p \geq 3$, une carte planaire dont toutes les faces ont le même degré p s'appelle une p-angulation. Dans les cas particuliers $p = 3$ et $p = 4$, on parle respectivement de triangulation et de quadrangulation.

Dans la suite on travaille avec des cartes enracinées. Une carte planaire m est enracinée si on distingue une arête (e_-^*, e_+^*) de m, qu'on appelle l'arête racine. Le sommet e_-^* s'appelle le sommet racine ou origine de la carte. L'enracinement permet de briser les symétries et rend l'étude des cartes techniquement plus facile.

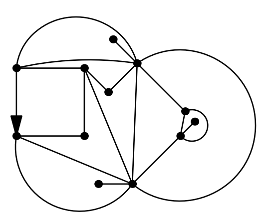

Figure 1.1.1 – Un exemple de quadrangulation enracinée à 10 faces.

Definition 1.1.2. *Soient u et v deux sommets d'une carte m. La distance de graphe $d_{gr}(u, v)$ est le nombre minimal d'arêtes nécessaires pour relier u et v dans m. On note $V(m)$ l'ensemble des sommets de m. Alors $(V(m), d_{gr})$ est un espace métrique compact.*

1.1.2 Historique

Nous présentons ensuite un bref historique des idées et des résultats sur les cartes.

Dans les années 1960, c'est Tutte qui a été l'initiateur des travaux sur les cartes [68, 69, 70, 71]. Il cherche à dénombrer les cartes, dans le but de démontrer le théorème des quatre couleurs. Le théorème des quatre couleurs énonce qu'il est possible de colorer les faces de n'importe quelle carte planaire, en n'utilisant que quatre couleurs différentes, de sorte que deux faces ayant une arête en commun soient toujours de deux couleurs distinctes. Tutte obtient de nombreuses formules combinatoires à l'aide de calculs sur les fonctions génératrices.

Il est aussi possible d'énumérer les cartes avec des intégrales matricielles. Cette méthode apparaît dans les travaux de 't Hooft dans les années 70 [67] et a donné lieu à de nombreux travaux sur ces thèmes, spécialement en lien avec la physique. On peut citer par exemple le livre de Lando et Zvonkine [38] pour voir le lien entre les cartes et les intégrales matricielles utilisant la formule de Wick. Les cartes sont utilisées en physique théorique comme modèle de géométrie aléatoire, en lien avec la gravité quantique en dimension 2. On renvoie ici aux travaux de Bouttier et Guitter, par exemple [18] et à la thèse de Bouttier [16]. Pour les relations entre les cartes et le champ libre gaussien, on pourra se référer aux travaux de Duplantier et Sheffield [28, 29].

Les méthodes énumératives de Tutte apportent des formules assez simples, en particulier on trouve que le nombre de quadrangulations enracinées à n faces est donné par

$$\mathrm{Card}\mathcal{Q}_n = \frac{2}{n+2} 3^n \frac{1}{n+1}\binom{2n}{n} \tag{1.2}$$

où on retrouve le n-ième nombre de Catalan $\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$. Ce nombre de Catalan est particulièrement connu pour donner le nombre d'arbres enracinés à n arêtes. Ces résultats peuvent être retrouvés à l'aide de bijections combinatoires. Les travaux de Cori et Vauquelin [21] puis de Schaeffer [65] permettent de relier les cartes planaires à différentes familles d'arbres. La bijection de Cori-Vauquelin-Schaeffer montre que les quadrangulations à n faces peuvent être encodées par des arbres à n arêtes étiquetés et permet donc de retrouver la formule (1.2) plus facilement. On remarque aussi que les quadrangulations à n faces sont également en bijection avec les cartes planaires générales (sans aucune condition sur le degré des faces) à n arêtes. En effet, si dans chaque face d'une carte générale à n arêtes on place un point et on le relie à chaque sommet de cette face, alors les nouvelles arêtes tracées forment une quadrangulation à n faces.

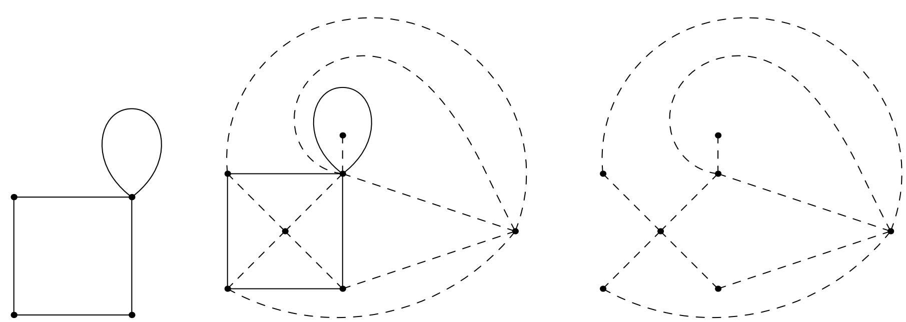

Figure 1.1.2 – Une carte générale à 5 arêtes et sa quadrangulation associée (en pointillés) à 5 faces.

Ainsi la formule (1.2) permet aussi de dénombrer les cartes générales à n arêtes.

Une généralisation très importante de la bijection précédente est la bijection de Bouttier-di Francesco-Guitter (voir [17]) qui sera d'une grande utilité dans notre travail et sera présentée en détail dans la suite.

1.1.3 Limite d'échelle

Une des idées majeures dans l'étude des cartes consiste à s'intéresser aux limites d'échelle, c'est-à-dire aux limites de grandes cartes convenablement renormalisées. Le but est d'obtenir un analogue en dimension 2 du théorème de Donsker qui fait apparaître le mouvement brownien comme limite d'échelle de marches aléatoires. L'objet limite en dimension 2 a été nommé dans un premier temps par Marckert et Mokkadem [53] et s'appelle la carte brownienne. Pour considérer les limites d'échelle de cartes, on a besoin de définir une distance entre deux espaces métriques compacts : il s'agit de la distance de Gromov-Hausdorff. On rappelle d'abord la définition de la distance de Hausdorff d_{Haus}. Si K_1, K_2 sont deux compacts dans un espace métrique (E,d), alors la distance d_{Haus} entre K_1 et K_2 est définie par

$$d_{\text{Haus}}(K_1,K_2) = \inf\{\epsilon > 0, K_1 \subset U_\epsilon(K_2) \text{ et } K_2 \subset U_\epsilon(K_1)\}$$

où $U_\epsilon(K_1) = \{x \in E, d(x,K_1) < \epsilon\}$ est le voisinage de rayon ϵ de K_1. Pour la distance de Gromov-Hausdorff d_{GH}, on plonge les deux espaces métriques compacts considérés dans un grand espace métrique. La définition est la suivante. Soient E_1, E_2 deux espaces métriques compacts, alors

$$d_{\text{GH}}(E_1,E_2) = \inf\{d_{\text{Haus}}(\Psi_1(E_1),\Psi_2(E_2))\}$$

où l'infimum porte sur tous les plongements isométriques $\Psi_1 : E_1 \to E$ et $\Psi_2 : E_2 \to E$ de E_1 et E_2 dans le même espace métrique E. L'ensemble des classes d'isométrie d'espaces métriques compacts muni de la distance de Gromov-Hausdorff d_{GH} est un espace polonais (voir [19]).

Le théorème suivant expose la convergence de certaines p-angulations vers la carte brownienne, au sens des limites d'échelle. On note $\mathbf{M}_n^p$ l'ensemble des p-angulations enracinées à n faces.

Theorem 1.1.3 (Le Gall [48], Miermont [58]). *On considère une carte $\mathcal{M}_n$ qui suit la loi uniforme sur l'ensemble $\mathbf{M}_n^p$. On suppose $p = 3$ ou $p \geq 4$ pair. On a la convergence en loi suivante au sens de Gromov-Hausdorff.*

$$\left(V(\mathcal{M}_n), c_p n^{-\frac{1}{4}} d_{gr}\right) \underset{n\to\infty}{\longrightarrow} (\mathbf{m}_\infty, D^*)$$

avec $c_3 = 6^{\frac{1}{4}}$ et $c_p = \left(\frac{9}{p(p-2)}\right)^{\frac{1}{4}}$ pour p pair, et où $(\mathbf{m}_\infty, D^*)$ est la carte brownienne.

Le cas $p = 4$ du théorème a été obtenu par Miermont [58] et indépendamment Le Gall [48] a établi la forme plus générale ci-dessus.

1.1.4 Définition de la carte brownienne

On donne une définition rapide de la carte brownienne [48]. Pour ce faire, on a d'abord besoin d'introduire l'arbre réel continu, qu'on appelle le CRT (Continuum Random Tree), décrit par Aldous dans [4, 5, 6]. Soit $(\mathbf{e}_s)_{0\leq s\leq 1}$ une excursion brownienne normalisée. Pour $s,t \in [0,1]$, on pose

$$d_{\mathbf{e}}(s,t) = \mathbf{e}_s + \mathbf{e}_t - 2\min\{\mathbf{e}_r : s\wedge t \leq r \leq s\vee t\}.$$

Remarquons que $d_{\mathbf{e}}$ définit une pseudo-distance sur $[0,1]$. On considère alors la relation d'équivalence définie pour $s,t \in [0,1]$ par

$$s \sim_{\mathbf{e}} t \text{ ssi } d_{\mathbf{e}}(s,t) = 0.$$

Le CRT est défini comme l'espace quotient $[0,1]/\sim_{\mathbf{e}}$ pour la relation d'équivalence $\sim_{\mathbf{e}}$, muni de la distance $d_{\mathbf{e}}$, et on le note $\mathcal{T}_{\mathbf{e}}$. On note aussi $p_{\mathbf{e}}$ la projection canonique de l'espace $[0,1]$ sur $\mathcal{T}_{\mathbf{e}}$.

Nous appelons $V = (V_s)_{0\le s\le 1}$ la tête du serpent brownien dirigée par $\mathbf{e}$. Rapidement, on peut décrire V, conditionnellement à $\mathbf{e}$, comme un processus gaussien centré à trajectoires continues de covariance donnée par

$$E(V_sV_t|\mathbf{e}) = \min_{s\wedge t\le r\le s\vee t} \mathbf{e}_r.$$

On ne décrit pas plus précisément le serpent brownien pour l'instant, car on en donnera une définition développée et de nombreuses propriétés dans la deuxième partie de cette thèse. On mentionne que $V_s = V_t$ pour tous $s,t \in [0,1]$ tels que $d_{\mathbf{e}}(s,t) = 0$, p.s. Ainsi, on peut exprimer V comme un processus indexé par $\mathcal{T}_{\mathbf{e}}$, et on a $V_s = V_{p_{\mathbf{e}}(s)}$ pour $s \in [0,1]$. Dans la suite, on pourra noter $V_s = V_a$ si $a = p_{\mathbf{e}}(s)$. Intuitivement, V est le mouvement brownien indexé par le CRT. On peut montrer que le processus $a \mapsto V_a$ défini sur $\mathcal{T}_{\mathbf{e}}$ muni de la distance $d_{\mathbf{e}}$ est $\frac{1}{2} - \epsilon$-höldérien pour tout $\epsilon \in [0, \frac{1}{2}[$.

On peut maintenant donner la définition de la carte brownienne, comme un espace quotient du CRT. Pour $s,t \in [0,1]$ tels que $s \le t$, on pose

$$D^0(s,t) = D^0(t,s) = V_s + V_t - 2\max(\min\{V_r : r \in [s,t]\}, \min\{V_r : r \in [0,s]\cup[t,1]\})$$

et pour $a,b \in \mathcal{T}_{\mathbf{e}}$,

$$D^0(a,b) = \min\{D^0(s,t) : (s,t) \in [0,1]^2, p_{\mathbf{e}}(s) = a, p_{\mathbf{e}}(t) = b\}.$$

Finalement, pour $a,b \in \mathcal{T}_{\mathbf{e}}$, on introduit

$$D^*(a,b) = \inf\{\sum_{i=1}^{k} D^0(a_{i-1}, a_i)\},$$

où l'infimum est pris sur tous les $k \ge 1$ et toutes les suites $(a_1,\ldots,a_k)$ d'éléments de $\mathcal{T}_{\mathbf{e}}$ telles que $a_0 = a$ et $a_k = b$. D^* ainsi définie est une pseudo-distance sur le CRT $\mathcal{T}_{\mathbf{e}}$, qui vérifie $D^* \le D^0$. On peut aussi interpréter D^* comme une fonction sur $[0,1]^2$ en posant $D^*(s,t) = D^*(p_{\mathbf{e}}(s), p_{\mathbf{e}}(t))$ pour $(s,t) \in [0,1]^2$. On peut alors poser, pour $a,b \in \mathcal{T}_{\mathbf{e}}$,

$$a \simeq b \text{ ssi } D^*(a,b) = 0$$

et $\simeq$ est une relation d'équivalence sur $\mathcal{T}_{\mathbf{e}}$. On note

$$\mathbf{m}_\infty = \mathcal{T}_{\mathbf{e}}/\simeq$$

l'espace quotient du CRT par la relation d'équivalence $\simeq$ et $\Pi : \mathcal{T}_{\mathbf{e}} \to \mathbf{m}_\infty$ la projection canonique. La carte brownienne correspond à cet espace $\mathbf{m}_\infty$ muni de la distance induite par D^*.

1.1.5 Quelques propriétés de la carte brownienne

Après de nombreux travaux, la carte brownienne apparaît comme un objet universel. En effet, elle se présente comme limite d'échelle de beaucoup de modèles de cartes. Citons les résultats de Le Gall et Beltran [9] pour les quadrangulations sans sommet pendant, Addario-Berry et Albenque [3] pour les triangulations et quadrangulations simples (c'est-à-dire sans boucle ni arête multiple), Bettinelli, Jacob et Miermont [11] pour les cartes générales, et ceux annoncés par Addario-Berry et Albenque pour les p-angulations avec p impair. Dans cette thèse on s'est intéressé au modèle des cartes biparties, qui sera expliqué en détail dans la suite.

Mentionnons maintenant quelques propriétés de la carte brownienne. Le théorème suivant donne sa dimension [47, Théorème 6.1].

Theorem 1.1.4. *La dimension de Hausdorff de la carte brownienne* $(\mathbf{m}_\infty, D^*)$ *est presque sûrement égale à* 4.

L'autre résultat que nous énonçons concerne sa topologie [50, Théorème 1.1].

Theorem 1.1.5. *La carte brownienne* $(\mathbf{m}_\infty, D^*)$ *est presque sûrement homéomorphe à la sphère* $\mathbb{S}^2$ *de* $\mathbb{R}^3$.

1.1.6 Limite locale

Il existe aussi un autre point de vue pour l'étude des limites des cartes planaires. À partir du travail d'Angel et Schramm [8], on considère des cartes dont la taille tend vers l'infini sans faire de changement d'échelle et on étudie la convergence en loi des boules autour de la racine. On parle de limite locale. On définit alors des cartes infinies enracinées qu'on peut voir comme des réseaux infinis aléatoires. On peut citer les travaux de Angel et Schramm [8] qui ont défini la limite locale des triangulations, appelée UIPT (Uniform Infinite Planar Triangulation), et ceux de Krikun [37], Chassaing et Durhuus [20] qui ont défini l'UIPQ (Uniform Infinite Planar Quadrangulation). On donne ici une définition rapide de cet objet. On note $\mathbf{Q}_n$ l'ensemble des quadrangulations enracinées à n faces et $\mathbf{Q}_f = \bigcup_{n\geq 1} \mathbf{Q}_n$ l'ensemble des quadrangulations enracinées finies.

Pour $r \geq 1$ et $q \in \mathbf{Q}_f$, la boule de rayon r notée $B_r(q)$ est la carte formée par les faces qui ont au moins un sommet à distance strictement inférieure à r de la racine. On introduit ensuite une distance locale d_{loc} entre deux quadrangulations. Pour $q, q' \in \mathbf{Q}_f$,

$$d_{\mathrm{loc}}(q, q') = (1 + \sup\{r \geq 1, B_r(q) = B_r(q')\})^{-1}$$

L'espace $(\mathbf{Q}_f, d_{\mathrm{loc}})$ est un espace métrique, et sa complétion est notée $(\mathbf{Q}, d_{\mathrm{loc}})$. On pose $\mathbf{Q}_\infty = \mathbf{Q} \setminus \mathbf{Q}_f$.

Theorem 1.1.6. *Soit* $\mathcal{Q}_n$ *une quadrangulation aléatoire de loi uniforme sur l'ensemble fini des quadrangulations enracinées à* n *faces. Alors*

$$\mathcal{Q}_n \underset{n\to\infty}{\longrightarrow} \mathcal{Q}_\infty$$

en loi pour d_{loc}. $\mathcal{Q}_\infty$ *est une quadrangulation uniforme infinie, on l'appelle l'UIPQ.*

La carte $\mathcal{Q}_\infty$ peut être plongée dans le plan de sorte qu'un compact du plan rencontre un nombre fini de faces de $\mathcal{Q}_\infty$.

1.1.7 Cartes biparties

Dans cette thèse, on s'est intéressé aux limites d'échelle des cartes biparties et on a prouvé un théorème de convergence vers la carte brownienne. Rappelons d'abord la définition d'une carte bipartie.

Definition 1.1.7. *Une carte planaire est bipartie si on peut colorer ses sommets en deux couleurs, de telle sorte que deux sommets de la même couleur ne sont pas reliés par une arête.*

Dans le cas planaire, c'est équivalent au fait que toutes les faces de la carte ont un degré pair. On note $\mathbf{M}_n^b$ l'ensemble des cartes biparties enracinées à n arêtes.

Voici le résultat principal obtenu.

Théorème A (Theorem 2.1.1)**.**

On considère une carte $\mathcal{M}_n$ qui suit la loi uniforme sur $\mathbf{M}_n^b$. Alors on a la convergence suivante au sens de Gromov-Hausdorff.

$$(V(\mathcal{M}_n), 2^{-1/4}n^{-1/4}d_{\mathrm{gr}}) \underset{n\to\infty}{\longrightarrow} (\mathbf{m}_\infty, D^*)$$

où $(\mathbf{m}_\infty, D^*)$ est la carte brownienne.

Pour prouver le Théorème A, on démontre d'abord un résultat de convergence pour les arbres, et on utilise la bijection de Bouttier di Francesco Guitter (qu'on notera bijection BDG) qui associe à une carte bipartie à n arêtes un arbre à deux types étiqueté à n arêtes. Avant d'expliquer en détail cette bijection, il est nécessaire de faire quelques rappels sur les arbres.

1.1.8 Arbres

On pose $\mathbb{N} = \{1, 2, \dots\}$ et par convention on a $\mathbb{N}^0 = \{\emptyset\}$. On introduit l'ensemble

$$\mathcal{U} = \bigcup_{n=0}^{\infty} \mathbb{N}^n.$$

Un élément de l'ensemble $\mathcal{U}$ est une suite $u = (u^1, \dots, u^n)$ d'éléments de $\mathbb{N}$ et on pose $|u| = n$ de telle sorte que $|u|$ corresponde à la "génération" de u. Si $u = (u^1, \dots, u^n)$ et $v = (v^1, \dots v^m)$ sont deux éléments de $\mathcal{U}$, alors $uv = (u^1, \dots u^n, v^1, \dots v^m)$ est la concaténation de u et v. On définit l'application $\pi : \mathcal{U} \setminus \{\emptyset\} \to \mathcal{U}$ par $\pi((u^1, \dots, u^n)) = (u^1, \dots, u^{n-1})$, on dit que $\pi(u)$ est le parent de u, et que u est un enfant de $\pi(u)$. On peut alors donner la définition d'un arbre planaire T,

Definition 1.1.8. *Un arbre planaire T est un sous-ensemble fini de $\mathcal{U}$ qui vérifie les trois propriétés suivantes.*

(i) $\emptyset \in T$;
(ii) *si* $u \in T \setminus \emptyset$, *alors* $\pi(u) \in T$;
(iii) *pour tout* $u \in T$, *il existe un entier* $k_u(T) \geq 0$ *tel que, pour tout* $j \in \mathbb{N}$, $uj \in T$ *si et seulement si* $1 \leq j \leq k_u(T)$.

$\emptyset$ s'appelle la racine de l'arbre, et dans (iii), le nombre $k_u(T)$ s'interprète comme le nombre d'enfants de u dans l'arbre T. On appelle taille de l'arbre T la quantité $|T| = \#T - 1$, qui correspond au nombre d'arêtes de T. On note $\mathbf{A}$ l'ensemble des arbres planaires. Si $u \in T$, $\sigma_u(T) = \{v \in \mathcal{U} : uv \in T\}$ est le sous-arbre de T issu de u.

On considère un arbre T et $n = |T|$ sa taille. T peut être codé par sa suite de contour $(u_0, \dots u_{2n})$ définie comme suit par récurrence : $u_0 = \emptyset$ et pour $i \in \{0, \dots, 2n-1\}, u_{i+1}$ est soit le premier enfant de u_i qui n'est pas encore apparu dans la suite $(u_0, \dots, u_i)$, ou le parent de u_i si tous les enfants de u_i sont déjà parmi $(u_0, \dots, u_i)$. On remarque que $u_{2n} = \emptyset$ et que tous les sommets de T apparaissent dans la suite de contour, certains plusieurs fois.

On donne ici la définition des arbres de Galton-Watson. Soit μ une mesure de probabilité sur $\mathbb{N}$ vérifiant $\mu(1) < 1$ et telle que $\sum k\mu(k) \leq 1$.

Definition 1.1.9. *On note* Π_μ *la loi d'un arbre de Galton-Watson de loi de reproduction* μ, *qui est la mesure de probabilité sur* $\mathbf{A}$ *vérifiant les propriétés suivantes.*
(i) *Pour tout* $k \geq 0$, $\Pi_\mu(k_\emptyset(T) = k) = \mu(k)$;
(ii) *pour* $k \geq 1$ *tel que* $\mu(k) > 0$, *sous* $\Pi_\mu(dT | k_\emptyset(T) = k)$, *les sous-arbres de* T *issus de* $1, \dots, k$ *sont i.i.d et de loi* Π_μ.

Autrement dit, un arbre de Galton–Watson de loi de reproduction μ est un arbre planaire aléatoire tel que chaque sommet de l'arbre ait un nombre d'enfants de loi μ et que les nombres d'enfants de sommets différents soient des variables aléatoires indépendantes. Quand la moyenne de μ vaut 1, on dit que l'arbre est critique.

Dans notre cas particulier des cartes biparties, nous avons besoin de considérer des arbres à deux types. On dira que les sommets de T à génération paire sont blancs et que les sommets de T à génération impaire sont noirs. On appelle respectivement T^0 et T^1 les ensembles des sommets blancs et noirs de T. Pour les arbres de Galton-Watson à deux types, on a deux lois de reproduction, une pour les sommets blancs et une pour les sommets noirs.

Nous introduisons maintenant les arbres étiquetés, dans le cas des arbres à deux types que nous considérons. Un arbre étiqueté est un couple $(T, (\ell(u))_{u \in T^0})$ tel que T est un arbre planaire et $(\ell(u))_{u \in T^0}$ est une suite d'étiquettes affectées aux sommets blancs de T, qui doit vérifier les propriétés suivantes
(i) Pour tout $u \in T^0$, $\ell(u) \in \mathbb{Z}$;
(ii) soient $v \in T^1$ et $k = k_v(T)$. On note $v_1 = v1, \dots, v_k = vk$ les enfants de v, et $v_0 = v_{k+1} = \pi(v)$. Alors, pour tout $i \in \{0, \dots, k\}$, $\ell(v_{i+1}) \geq \ell(v_i) - 1$.

L'entier $\ell(u)$ s'appelle l'étiquette de u. La propriété (ii) signifie que si on tourne autour d'un sommet noir dans le sens horaire, les étiquettes i et j de deux sommets blancs successifs voisins de v vérifient $j \geq i - 1$. On notera $\mathbf{T}_n$ l'ensemble des arbres à deux types étiquetés à n arêtes.

1.1.9 Fonctions de contour et d'étiquette

On présente un couple de fonctions qui permet de coder un arbre étiqueté $(T, (\ell(u))_{u\in T^0})$. Rappelons d'abord que si $|T| = n$, alors on a la suite de contour $(u_0, \dots, u_{2n})$ de T. On peut remarquer que u_i est un sommet blanc si i est pair, et un sommet noir si i est impair. On définit alors pour $0 \le i \le 2n$

$$C_i^T = |u_i|.$$

On prolonge ensuite C^T à l'intervalle $[0, 2n]$ par interpolation linéaire, et on obtient la fonction de contour de l'arbre T.

Pour $0 \le i \le n$, on pose $v_i = u_{2i}$. La suite $(v_0, \dots, v_n)$ s'appelle la suite de contour blanche de l'arbre T. On introduit

$$C_i^{T^0} = \frac{1}{2}|v_i|$$

et

$$L_i^{T^0} = \ell(v_i).$$

On voit que C^{T^0} est liée à C^T par la relation $C_i^{T^0} = \frac{1}{2}C_{2i}^T$. On prolonge également C^{T^0} et L^{T^0} par interpolation linéaire, et ces fonctions sont alors définies sur l'intervalle $[0, n]$. On appelle C^{T^0} la fonction de contour blanche de T et L^{T^0} la fonction d'étiquette.

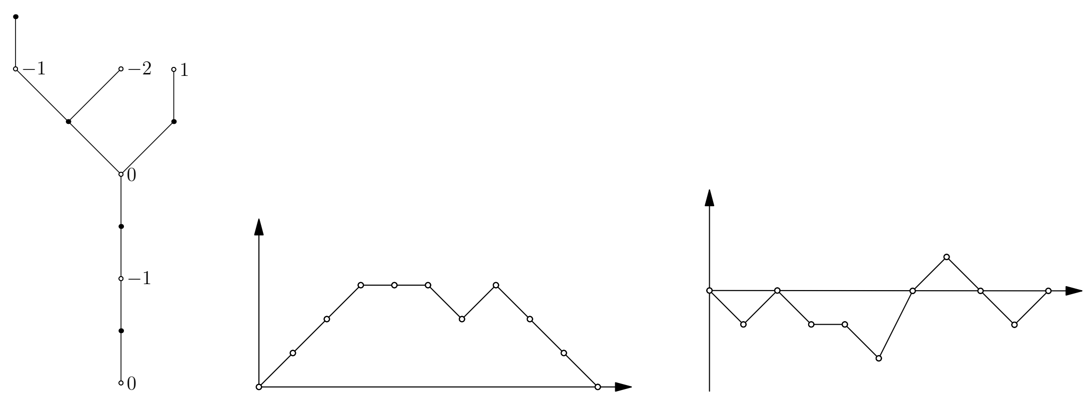

Figure 1.1.3 – Un arbre T à $n = 10$ arêtes, sa fonction de contour blanche C^{T^0} et sa fonction d'étiquette L^{T^0}.

Il est alors assez facile de vérifier que le couple (C^T, L^{T^0}) détermine de manière unique l'arbre étiqueté $(T, (\ell(u))_{u\in T^0})$. Au contraire, le couple (C^{T^0}, L^{T^0}) ne permet pas d'avoir assez d'information pour retrouver l'arbre.

1.1.10 Bijection BDG

Nous exposons la bijection BDG de Bouttier, Di Francesco et Guitter, dans le cas particulier des cartes biparties qui nous intéresse. Pour une version plus générale de cette bijection et pour les preuves, on peut se référer à [17]. Nous rappelons que $\mathbf{M}_n^b$ est l'ensemble des cartes planaires biparties enracinées à n arêtes. On introduit ici l'ensemble $\mathbf{M}_n^{b\bullet}$ des cartes biparties à n arêtes enracinées et pointées, ce qui signifie qu'on a distingué un sommet particulier dans la carte qu'on appelle sommet pointé. La bijection BDG fait correspondre les ensembles $\mathbf{T}_n \times \{0, 1\}$ et $\mathbf{M}_n^{b\bullet}$.

On part d'abord d'un arbre à deux types étiqueté à n arêtes pour arriver à une carte bipartie enracinée et pointée à n arêtes. Soit $(T, (\ell(u))_{u \in T^0}) \in \mathbf{T}_n$ et $\epsilon \in \{0, 1\}$. Comme précédemment, $(v_0, \ldots, v_n)$ est la suite de contour blanche de T. On appelle coin de T un secteur angulaire autour d'un sommet, entre deux arêtes consécutives dans le sens horaire. On donne à chaque coin l'étiquette du sommet associé. Dans la suite de contour blanche $(v_0, \ldots, v_n)$, chaque v_i à part le dernier correspond à un coin de l'arbre T, on pourra donc nommer les coins v_i (avec un petit abus de langage). On ajoute un sommet supplémentaire ∂ dans le plan. On effectue la construction suivante, illustrée sur la figure 1.1.4. Pour tout $i \in \{0, \ldots n-1\}$,

— si $\ell(v_i) = \min\{\ell(v) : v \in T^0\}$, alors on trace une arête entre le coin v_i et ∂ ;

— si $\ell(v_i) > \min\{\ell(v) : v \in T^0\}$, alors on trace une arête entre le coin v_i et le prochain coin v_j dans le sens horaire dont l'étiquette est $\ell(v_i) - 1$.

Grâce à la propriété (ii) de la définition des arbres étiquetés, il est possible de faire cette construction de façon à ce que les arêtes tracées ne se croisent pas, et ne croisent pas les arêtes de l'arbre T. Les sommets de l'arbre, le sommet ∂ et les arêtes nouvellement dessinées forment une carte planaire bipartie, $M^\bullet$, à n arêtes. On dit alors que le sommet rajouté ∂ est le sommet pointé de la carte, et l'arête racine est la première arête qu'on dessine dans la construction. Le paramètre ϵ permet de choisir l'orientation de l'arête racine. Par convention, on dira que le sommet racine est $\emptyset$ ssi $\epsilon = 0$.

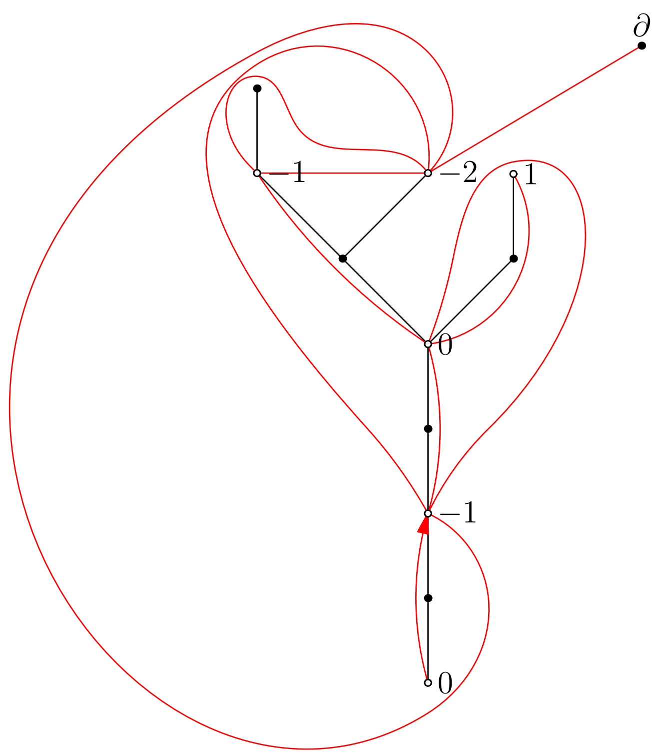

Figure 1.1.4 – Exemple de la construction de la bijection BDG en partant d'un arbre pour arriver à une carte.

On s'intéresse ensuite à la construction inverse, dont la figure 1.1.5 est un exemple. On part d'une carte bipartie à n arêtes, enracinée et pointée. On attribue des étiquettes à chaque sommet de la carte, l'étiquette d'un sommet étant sa distance de graphe au sommet pointé, qui est donc étiqueté 0. On regarde ensuite les étiquettes dans chaque face. Dans une face, dans le sens horaire, on a une suite d'entiers de longueur $2k$, dont les incréments sont $+1$ ou -1. Parmi les $2k$ sommets de la face, on choisit les k qui sont suivis par un sommet d'étiquette inférieure. On ajoute alors un sommet à l'intérieur de la face considérée, et on relie les k sommets choisis à ce sommet rajouté. Après avoir fait cette construction dans chaque face, on efface les arêtes initiales de la carte et le sommet pointé qui s'est retrouvé isolé. On obtient alors un arbre à deux types. Les sommets étiquetés de cet arbre sont les sommets de la carte (excepté ∂), et les sommets non étiquetés correspondent aux faces de la carte.

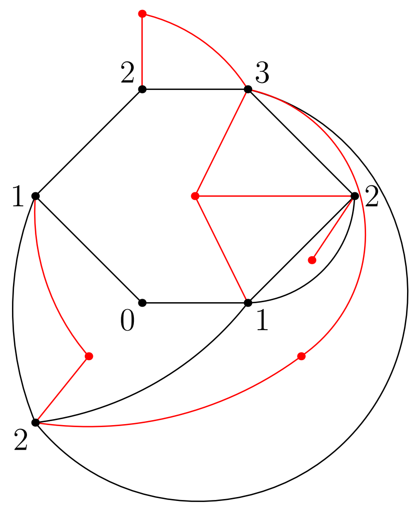

Figure 1.1.5 – Exemple de la construction de la bijection BDG en partant d'une carte bipartie pour arriver à un arbre.

Pour la preuve de cette bijection, on renvoie à [17]. Notons ici qu'on a la relation suivante entre les distances de graphe au sommet pointé dans la carte et les étiquettes dans l'arbre.

$$d_{\mathrm{gr}}^{M^\bullet}(\partial, u) = \ell(u) - \min\{\ell(v) : v \in T^0\} + 1.$$

1.1.11 Étapes de la preuve du résultat sur les cartes biparties

Les rappels sur les arbres et sur la bijection BDG nous permettent d'énoncer le résultat de convergence pour les arbres qui est une étape primordiale pour la preuve du Théorème A. Grâce à la bijection précédente, on obtient un arbre qui correspond à une carte choisie uniformément parmi les cartes biparties enracinées et pointées à n arêtes. Cet arbre est en fait un arbre de Galton-Watson à deux types, conditionné à avoir n arêtes, dont les lois de reproduction pour les sommets blancs et noirs peuvent être calculées et sont données respectivement par $\mu_0(k) = \frac{2}{3}\left(\frac{1}{3}\right)^k$ et $\mu_1(k) = \frac{3}{8}\binom{2k+1}{k}\left(\frac{3}{16}\right)^k$ pour $k \geq 0$. On considère maintenant un tel arbre de Galton-Watson à deux types $(\mathcal{T}, (\ell(u))_{u\in\mathcal{T}^0})$, et on pose $N = |\mathcal{T}|$. On a alors

Théorème B (Theorem 2.4.1).

La loi conditionnelle du couple

$$\left(n^{-1/2}C^{\mathfrak{T}^0}_{nt}, n^{-1/4}L^{\mathfrak{T}^0}_{nt}\right)_{0\le t\le 1}$$

des fonctions de contour et d'étiquette, sachant que $N = n$, converge quand $n \to \infty$ vers la loi de

$$\left(\frac{4\sqrt{2}}{9}\mathbf{e}_t, 2^{1/4}V_t\right)_{0\le t\le 1}$$

où on rappelle que $\mathbf{e}$ est une excursion brownienne normalisée et V est la tête du serpent brownien dirigé par cette excursion.

Pour montrer ce résultat de convergence pour les arbres, deux points-clés apparaissent. Le premier est l'utilisation d'un argument d'absolue continuité. Au lieu de conditionner par $N = n$, on conditionne par $N > (1-\delta)n$ et on utilise des résultats de convergence déjà existants de Miermont [57]. On fait ainsi apparaître la densité de la loi de l'excursion brownienne normalisée par rapport à la loi de l'excursion brownienne conditionnée à avoir une longueur supérieure à $1-\delta$. Le second est la construction d'une suite qui code l'arbre blanc et qui peut être vue comme un chemin de Lukasiewicz modifié. Cette suite a la loi d'une marche aléatoire dont on peut calculer la loi de saut. De plus, elle est reliée de façon assez simple à la fonction de contour blanche.

Pour passer du Théorème B de convergence pour les arbres au Théorème A de convergence vers la carte brownienne pour les cartes biparties, on a besoin d'arguments introduits par Le Gall [48, 9]. En particulier, on utilise un lemme crucial de réenracinement uniforme. On obtient un théorème de convergence proche du Théorème B, mais faisant intervenir aussi une troisième composante correspondant aux distances entre deux sommets arbitraires de la carte. Le lemme dit que la limite obtenue pour cette troisième composante est en fait égale à la distance D^* introduite précédemment dans la construction de la carte brownienne.

Une dernière remarque importante est de souligner que, dans la bijection BDG et donc dans le Théorème B, on travaille avec des cartes biparties enracinées et pointées. Or le résultat souhaité est énoncé pour des cartes seulement enracinées. Pour passer des cartes enracinées et pointées aux cartes enracinées, on se sert d'un argument de dépointage, utilisé en particulier par Bettinelli, Jacob et Miermont dans [11].

1.2 Serpent brownien

1.2.1 Définition du serpent brownien

Le serpent brownien a été introduit par Le Gall et est un objet très utile qui apparaît dans de nombreuses théories. Pour une présentation générale et détaillée de cet objet, on renvoie à [44, 42]. On en donne ici la définition. Nous présentons d'abord l'espace des trajectoire arrêtées. Une trajectoire arrêtée w est une application continue définie sur un intervalle $[0, \zeta_{(w)}]$ à valeurs dans $\mathbb{R}^d$. On remarque ici que la trajectoire peut être à valeurs dans un espace plus général, mais dans le cadre de nos travaux on se limite au cas de $\mathbb{R}^d$ (voire $\mathbb{R}$). La quantité $\zeta_{(w)}$ s'appelle le temps de vie de la trajectoire w, et on note $\hat{w} = w(\zeta_{(w)})$ le point terminal de w.

On appelle $\mathcal{W}$ l'ensemble de toutes les trajectoires arrêtées. Pour $x \in \mathbb{R}^d$, $\mathcal{W}_x$ est l'ensemble de toutes les trajectoires arrêtées issues de x, c'est-à-dire telles que $w(0) = x$. Si $\zeta(w) = 0$, la trajectoire w est réduite à un point. Dans $\mathcal{W}_x$, on note cette trajectoire $\underline{x}$.

On peut définir la distance suivante sur l'espace $\mathcal{W}_x$.

$$d(w, w') = \sup_{t \geq 0} |w(t \wedge \zeta_{(w)}) - w'(t \wedge \zeta_{(w')})| + |\zeta_{(w)} - \zeta_{(w')}|.$$

L'espace $\mathcal{W}_x$ muni de la distance d est polonais.

La mesure de probabilité suivante va être très utile.

Definition 1.2.1. *Pour $w \in \mathcal{W}_x$, $a \in [0, \zeta_{(w)}]$ et $b \geq a$, on définit une unique mesure de probabilité sur $\mathcal{W}_x$, notée $R_{a,b}(w, dw')$, qui vérifie les trois propriétés suivantes.*

(i) $\zeta_{(w')} = b$, $R_{a,b}(w, dw')$ p.s. ;

(ii) $w'(t) = w(t)$ pour tout $t \in [0, a]$, $R_{a,b}(w, dw')$ p.s. ;

(iii) la loi sous $R_{a,b}(w, dw')$ de la trajectoire $(w'(a + t))_{t \geq 0}$ est celle d'un mouvement brownien dans $\mathbb{R}^d$ issu de $w(a)$ et arrêté au temps $b - a$.

Moins formellement, sous $R_{a,b}(w, dw')$, w' est une trajectoire arrêtée de temps de vie b qu'on obtient à partir de la trajectoire w en effaçant w après l'instant a et en plaçant entre les instants a et b un morceau de trajectoire brownienne dans $\mathbb{R}^d$.

On considère un mouvement brownien réfléchi $(\beta_s)_{s \geq 0}$ dans $\mathbb{R}_+$, issu de $\beta_0 = \lambda$. On appelle $\theta_s^\lambda(dadb)$ la loi du couple $(\inf_{0 \leq r \leq s} \beta_r, \beta_s)$. Grâce aux outils et aux définitions précédentes, on peut énoncer le résultat suivant qui donne la définition du serpent brownien.

Theorem 1.2.2. *Il existe un processus de Markov continu à valeurs dans $\mathcal{W}_x$, noté $(W_s)_{s \geq 0}$, issu de $W_0 = \underline{x}$, dont les noyaux de transition sont donnés par*

$$Q_s(w, dw') = \int \theta_s^{\zeta_{(w)}}(dadb) R_{a,b}(w, dw').$$

Le temps de vie de W_s est désigné par ζ_s. Le processus $(\zeta_s)_{s \geq 0}$ est alors un mouvement brownien réfléchi dans $\mathbb{R}_+$ issu de 0, et il est markovien par rapport à la filtration naturelle de $(W_s)_{s \geq 0}$. Pour $\delta > 0$, l'application $s \mapsto W_s$ est p.s. localement hölderienne d'exposant $\frac{1}{4} - \delta$.

Le processus $(W_s)_{s \geq 0}$ s'appelle le serpent brownien, et $(\hat{W}_s)_{s \geq 0}$ désigne la tête du serpent brownien.

De plus, $(W_s)_{s \geq 0}$ est un processus fortement markovien. Remarquons aussi que $(W_s)_{s \geq 0}$ peut être vu comme la limite d'une chaîne de Markov discrète $(W_k^\epsilon)_{k \geq 0}$ convenablement changée de temps, quand $\epsilon \to 0$. Notons $\mathcal{W}_x^\epsilon$ l'ensemble des trajectoires arrêtées issues de x dont le temps de vie appartient à $\epsilon\mathbb{Z}_+$. La chaîne de Markov $(W_k^\epsilon)_{k \geq 0}$ est construite à partir du noyau de transition Π^ϵ suivant de $\mathcal{W}_x^\epsilon$ dans $\mathcal{W}_x^\epsilon$.

— Si $\zeta_{(w)} \geq \epsilon$, alors

$$\Pi^\epsilon(w, dw') = \frac{1}{2} R_{\zeta_{(w)}, \zeta_{(w)} + \epsilon}(w, dw') + \frac{1}{2} R_{\zeta_{(w)} - \epsilon, \zeta_{(w)} - \epsilon}(w, dw');$$

— si $\zeta_{(w)} = 0$, alors
$$\Pi^\epsilon(w, dw') = R_{0,\epsilon}(w, dw').$$

On peut appeler $(W_k^\epsilon)_{k\geq 0}$ le serpent discret. On mentionne également que le processus des temps de vie $(\zeta_k^\epsilon)_{k\geq 0}$ de $(W_k^\epsilon)_{k\geq 0}$ est une marche aléatoire simple sur $\epsilon\mathbb{Z}_+$.

1.2.2 Excursions pour le mouvement brownien

La seconde partie de cette thèse a été consacrée à l'étude des excursions du serpent brownien. Nous donnons dans un premier temps des éléments de théorie des excursions pour le mouvement brownien, afin de pouvoir mieux comprendre nos résultats sur le serpent. On peut se référer à [63, 64]. Dans cette partie on considère un mouvement brownien réel $(B_t)_{t\geq 0}$ issu de 0. On appelle $(\mathcal{F}_t)_{t\geq 0}$ sa filtration naturelle, que l'on complète par les négligeables de $\mathcal{F}_\infty$. On rappelle d'abord la définition du temps local, qui est donnée par la proposition suivante.

Proposition 1.2.3. *Il existe un processus aléatoire à valeurs dans $\mathbb{R}_+$, noté $(L_t^a)_{a\in\mathbb{R}, t\geq 0}$, unique à indistinguabilité près, vérifiant*
(i) p.s. la fonction $(t, a) \mapsto L_t^a$ est continue et croissante en t ;
(ii) p.s. pour toute fonction mesurable $\phi : \mathbb{R} \to \mathbb{R}_+$ et pour tout $t \geq 0$,
$$\int_0^t \phi(B_s)ds = \int da\phi(a)L_t^a.$$
L_t^a s'appelle le temps local du mouvement brownien B au niveau a et à l'instant t.

De plus, pour tout $\delta > 0$, la fonction $(t, a) \mapsto L_t^a$ est p.s. localement hölderienne d'exposant $\frac{1}{2} - \delta$. On peut maintenant définir l'inverse continu à droite du temps local en 0 par
$$\tau_s = \inf\{t \geq 0 : L_t^0 > s\}.$$
Sa limite à gauche est notée
$$\tau_{s^-} = \inf\{t \geq 0 : L_t^0 = s\}.$$
On nomme D l'ensemble dénombrable des instants de discontinuité de $s \mapsto \tau_s$. La proposition suivante va permettre de définir les excursions du mouvement brownien B.

Proposition 1.2.4. *Pour $u \in D$, les intervalles $]\tau_{u^-}, \tau_u[$ sont exactement les intervalles d'excursion de B en dehors de 0. Autrement dit, les composantes connexes de $\mathbb{R}_+ \setminus \{t \geq 0 : B_t = 0\}$ sont les $]\tau_{u^-}, \tau_u[$ pour $u \in D$.*

Pour $u \in D$ et $t \geq 0$, on peut alors poser
$$e_u(t) = B_{(\tau_{u^-}+t)\wedge\tau_u}.$$
Ainsi définie, e_u est une fonction continue sur $\mathbb{R}_+$, à valeurs dans $\mathbb{R}$, nulle sauf sur $]0, \sigma(e_u)[$ où $\sigma(e_u) = \tau_u - \tau_{u^-}$.

Definition 1.2.5. *La mesure ponctuelle des excursions de B est donnée par*
$$N = \sum_{u\in D} \delta_{(u,e_u)}.$$

Le résultat suivant permet de décrire le processus des excursions de B. Rappelons qu'un processus de Poisson ponctuel sur F est une mesure de Poisson sur $\mathbb{R}_+ \times F$ d'intensité $dt \otimes \nu(dx)$, où ν est appelée la mesure caractéristique du processus. On note $C(\mathbb{R}_+, \mathbb{R})$ l'espace des fonctions continues de $\mathbb{R}_+$ dans $\mathbb{R}$.

Theorem 1.2.6. *N est un processus de Poisson ponctuel sur $C(\mathbb{R}_+, \mathbb{R})$.*

La mesure caractéristique de N est notée $n(de)$ et s'appelle mesure d'Itô des excursions du mouvement brownien. n est une mesure σ-finie. De plus, elle peut s'écrire $n = n_+ + n_-$, où n_+ est portée par l'ensemble des excursions positives, et n_- par l'ensemble des excursions négatives. On notera $e_u(t) = e(t)$.

Grâce à la propriété de changement d'échelle du mouvement brownien, on peut aussi définir les excursions du mouvement brownien dont la durée de vie est fixée. Plus précisément, il existe une unique famille de mesures de probabilité $(n_{(s)})_{s>0}$ telle que les propriétés suivantes soient vérifiées.

(i) $n_+ = \int_0^\infty \frac{ds}{2\sqrt{2\pi t^3}} n_{(s)}$;

(ii) pour tout $s > 0$, $n_{(s)}(\sigma = s) = 1$;

(iii) on a la propriété de scaling suivante : pour tous $\lambda > 0$ et $\sigma > 0$, la loi sous $n_{(s)}(de)$ de $e_\lambda(t) = \sqrt{\lambda} e(t/\lambda)$ est $n_{(\lambda s)}$.

La mesure de probabilité $n_{(1)}$ s'appelle la loi de l'excursion brownienne normalisée. Sous $n_{(1)}$, on note $e(t) = \mathbf{e}_t$.

On peut voir la mesure n_+ comme la limite quand $\epsilon \to 0$ de $\frac{1}{2\epsilon}$ fois la loi d'un mouvement brownien issu de ϵ et arrêté quand il touche 0. Plus précisément, on fixe $\delta > 0$. Rappelons que $T_0 = \inf\{t \geq 0 : B_t = 0\}$. Pour toute fonction ϕ continue bornée, nulle sur $\{e : \sup_{s \geq 0} e(s) < \delta\}$, on a

$$\frac{1}{2\epsilon} E_\epsilon \left(\phi(B_{t \wedge T_0}, t \geq 0) \right) \underset{\epsilon \to \infty}{\longrightarrow} n_+(\phi(e(t), t \geq 0)). \tag{1.3}$$

Ce résultat s'obtient comme une conséquence de

$$n_+ \left(\sup_{s \geq 0} e(s) > \epsilon \right) = \frac{1}{2\epsilon}$$

et de la propriété de Markov forte pour le mouvement brownien.

Enfin, on expose la décomposition de Bismut pour la mesure d'Itô n. On se réfère à [63, Chapter XII]. On pose $R(w) = \inf\{t > 0 : w(t) = 0\}$.

Theorem 1.2.7. *Soit $\bar{n}_+$ la mesure définie sur $C(\mathbb{R}_+, \mathbb{R})$ par*

$$\bar{n}_+(dt, de) = \mathbf{1}_{\{0 \leq t \leq R(e)\}} dt \; n_+(de).$$

La "loi" de $e(t)$ sous $\bar{n}_+$ est la mesure de Lebesque sur $\mathbb{R}_+$. De plus, sous $\bar{n}_+$, conditionnellement à $e(t) = a$, les processus $(e((t-s)^+))_{s \geq 0}$ et $(e(t+s))_{s \geq 0}$ sont deux mouvements browniens réels issus de a, indépendants, arrêtés en leur temps d'atteinte de 0.

1.2.3 Excursions du serpent brownien : mesure d'excursion $\mathbb{N}_x$

Dans cette partie on définit une mesure d'excursion pour le serpent brownien de manière analogue à la définition de la mesure d'Itô d'excursions pour le mouvement brownien décrite précédemment.

On travaille avec le processus $(W_s)_{s\geq 0}$ issu de $W_0 = \underline{x}$. Alors le processus des temps de vie $(\zeta_s)_{s\geq 0}$ est un mouvement brownien réfléchi issu de 0. On peut considérer le processus ponctuel des excursions de ζ. On le note $N = \sum_{u\in D} \delta_{(u,e_u)}$ comme dans la partie précédente, où D est l'ensemble des temps de discontinuité de l'inverse (τ_s) du temps local en 0 de (ζ_s). Les excursions de ζ en dehors de 0 correspondent exactement aux excursions de W en dehors de $\underline{x}$. On définit les excursions du serpent brownien en dehors de $\underline{x}$. Pour $u \in D$ et $s \geq 0$,

$$\omega_s^u = W_{(\tau_{u-}+s)\wedge\tau_u}.$$

La fonction ω^u est continue sur $\mathbb{R}_+$ à valeurs dans $\mathcal{W}_x$, et le temps de vie de ω_s^u est donné par $e_u(s)$.

Le théorème suivant est l'analogue du Théorème 1.2.6.

Theorem 1.2.8. *Le processus* $\sum_{u\in D} \delta_{(u,\omega^u)}$ *est un processus de Poisson ponctuel sur* $C(\mathbb{R}_+, \mathcal{W}_x)$.

On note $\mathbb{N}_x(d\omega)$ sa mesure caractéristique, $\mathbb{N}_x$ est la mesure d'excursion pour le serpent brownien. La mesure $\mathbb{N}_x$ est reliée à n_+. En effet, la mesure image de $\mathbb{N}_x(d\omega)$ par l'application $\omega \mapsto (\zeta_{\omega_s})_{s\geq 0}$ est la mesure $n_+(de)$. On a en particulier, pour tout $\epsilon > 0$,

$$\mathbb{N}_x\left(\sup_{s\geq 0} \zeta_s > \epsilon\right) = n_+\left(\sup_{s\geq 0} e(s) > \epsilon\right) = \frac{1}{2\epsilon}.$$

La mesure d'excursion $\mathbb{N}_x$ du serpent brownien va être très utile dans la suite. Sous $\mathbb{N}_x$, on note $\sigma = \sup\{s \geq 0 : \zeta_s > 0\}$ qui représente la durée de l'excursion. De manière analogue à la définition des mesures $n_{(s)}$, on peut considérer, pour tout $s > 0$, $\mathbb{N}_x^{(s)} = \mathbb{N}_x(.|\sigma = s)$. On note p_ζ la projection canonique de $[0, \sigma]$ sur l'arbre codé par le processus ζ.

1.2.4 Mesure de sortie et propriétés de $\mathbb{N}_x$

Nous allons définir la mesure de sortie du serpent brownien, et en exposer des propriétés importantes. On se réfère ici à [44, Chapter V]. Soit D un ouvert de $\mathbb{R}^d$ tel que $x \in D$. Pour $w \in \mathcal{W}_x$, on pose

$$\tau_D(w) = \inf\{t \in [0, \zeta_{(w)}] : w(t) \notin D\},$$

qui correspond au premier instant où w sort de D. On introduit maintenant

$$E^D = \{W_s(\tau_D(W_s)), s \geq 0, \tau_D(W_s) < \infty\}$$

l'ensemble de tous les points de sortie en dehors de D des trajectoires W_s, pour celles qui sortent de D. Le but est de construire $\mathbb{N}_x$ p.p. une mesure aléatoire qui est "uniforme" sur E^D. On a d'abord besoin de construire un processus continu croissant qui ne croît que sur l'ensemble $\{s \geq 0 : \tau_D(W_s) = \zeta_s\}$.

Proposition 1.2.9. *Sous $\mathbb{N}_x$, la formule suivante définit un processus continu croissant $(L_s^D)_{s\geq 0}$.*

$$L_s^D = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\int_0^s dr \mathbf{1}_{\{\tau_D(W_r)<\zeta_r<\tau_D(W_r)+\epsilon\}}.$$

On l'appelle temps local de sortie en dehors de D.

Il est alors possible de définir la mesure de sortie $\mathcal{Z}^D$ en dehors de D sous $\mathbb{N}_x$. Pour une fonction mesurable bornée g,

$$\langle \mathcal{Z}^D, g\rangle = \int_0^\sigma dL_s^D g(\hat{W}_s).$$

Un outil primordial dans les travaux sur le serpent brownien et pour la suite est la propriété de Markov spéciale. On peut remarquer que, $\mathbb{N}_x$ p.p., l'ensemble $\{s\geq 0 : \tau_D(W_s) < \zeta_s\}$ est ouvert, donc s'écrit comme une réunion disjointe d'ouverts $]a_i, b_i[, i \in I$. De plus, on a pour $i\in I$, pour tout $s \in]a_i, b_i[$,

$$\tau_D(W_s) = \tau_D(W_{a_i}) = \zeta_{a_i}.$$

Pour $s \in]a_i, b_i[$, les trajectoires W_s coïncident jusqu'à ce qu'elles sortent de D. Pour $i \in I$, on définit $W^{(i)}$ par

$$W_s^{(i)}(t) = W_{(a_i+s)\wedge b_i}(\zeta_{a_i} + t)$$

pour $s \geq 0$ et $0 \leq t \leq \zeta_{(a_i+s)\wedge b_i} - \zeta_{a_i}$. En fait, les $W^{(i)}$ ainsi construits correspondent aux excursions du serpent brownien “en dehors” de D (ces excursions ne sont pas vraiment en dehors de D car elles partent d'un point sur le bord de D mais peuvent revenir à l'intérieur du domaine D). On pose

$$\eta_s^D = \inf\{r \geq 0 : \int_0^r du \mathbf{1}_{\{\zeta_u\leq\tau_D(W_u)\}} > s\}$$

et on appelle $\mathcal{E}^D$ la tribu engendrée par les trajectoires $(W_{\eta_s^D})_{s\geq 0}$ et par les négligeables pour $\mathbb{N}_x$. La tribu $\mathcal{E}^D$ contient l'information sur les W_s avant leur sortie de D. On montre que $\mathcal{Z}^D$ est $\mathcal{E}^D$-mesurable. Énonçons maintenant la propriété de Markov spéciale ([46]).

Proposition 1.2.10. *Sous la mesure d'excursion $\mathbb{N}_x$, conditionnellement à $\mathcal{E}^D$, la mesure ponctuelle*

$$\sum_{i\in I} \delta_{W^{(i)}}$$

est une mesure de Poisson d'intensité donnée par $\int_{\mathbb{R}^d} \mathcal{Z}^D(dy)\mathbb{N}_y$.

De nombreuses applications de cette propriété de Markov spéciale apparaîtront dans la suite.

On mentionne maintenant le lien entre les mesures d'excursion $\mathbb{N}_x$, la mesure de sortie $\mathcal{Z}^D$ et les équations différentielles. Le résultat suivant est souvent utile notamment pour calculer des lois sous $\mathbb{N}_x$.

Proposition 1.2.11. *Soient g une fonction mesurable positive bornée sur ∂D, et $x \in D$. On pose*

$$u(x) = \mathbb{N}_x(1 - \exp(-\langle \mathcal{Z}^D, g\rangle)).$$

Alors u est solution de $\Delta u = 4u^2$ dans D. Si de plus g est continue, alors u est solution de $\Delta u = 4u^2$ dans D avec conditions au bord $u_{|\partial D} = g$.

Finalement, on rappelle la propriété de scaling pour la mesure d'excursion $\mathbb{N}_x$. Considérons pour $s \geq 0$ et $\lambda > 0$ le processus W' défini par

$$W'_s(t) = \sqrt{\lambda} W_{s/\lambda^2}\left(\frac{t}{\lambda}\right)$$

pour $0 \leq t \leq \zeta'_s = \lambda \zeta_{s/\lambda^2}$.

Proposition 1.2.12. *Sous* $\mathbb{N}_x$, W' *a pour loi* $\lambda \mathbb{N}_{x\sqrt{\lambda}}$.

1.2.5 Lien avec le mouvement brownien indexé par un arbre

Nous exposons maintenant la construction d'un mouvement brownien indexé par un arbre réel, cet objet a des liens très forts avec le serpent brownien. Soit (T, d) un arbre réel déterministe dont la racine est notée ρ. On peut définir un processus gaussien centré $(V_u)_{u \in T}$ à valeurs dans $\mathbb{R}^d$ par $V_\rho = 0$ et pour $u, v \in T$,

$$\mathrm{Cov}(V_u, V_v) = d(\rho, u \wedge v) I,$$

où I est la matrice identité et $u \wedge v$ est le dernier ancêtre commun de u et v. Autrement dit, V_u est la position d'un mouvement brownien en dimension d à l'instant $d(\rho, u)$, V_v est la position d'un mouvement brownien en dimension d à l'instant $d(\rho, v)$, on considère le même mouvement brownien entre 0 et $d(\rho, u \wedge v)$ puis deux mouvements browniens indépendants respectivement entre $d(\rho, u \wedge v)$ et $d(\rho, u)$ et entre $d(\rho, u \wedge v)$ et $d(\rho, v)$.

Si comme arbre T on choisit le CRT d'Aldous $\mathcal{T}_{\mathbf{e}}$ introduit dans la partie précédente, la construction précédente donne un processus $V^{\mathcal{T}_{\mathbf{e}}}$ à trajectoires continues qui est, conditionnellement à $\mathcal{T}_{\mathbf{e}}$, un mouvement brownien indexé par $\mathcal{T}_{\mathbf{e}}$. Ici la définition de $V^{\mathcal{T}_{\mathbf{e}}}$ pose certaines difficultés techniques puisqu'on considère un processus indexé par un ensemble aléatoire.

Rappelons que $p_{\mathbf{e}}$ est la projection canonique de $[0, 1]$ sur $\mathcal{T}_{\mathbf{e}}$. Alors la loi de $(V^{\mathcal{T}_{\mathbf{e}}}_{p_{\mathbf{e}}(s)})_{0 \leq s \leq 1}$ coïncide avec la loi de $(\hat{W}_s)_{0 \leq s \leq 1}$ sous $\mathbb{N}_0^{(1)} = \mathbb{N}_0(.|\sigma = 1)$, ainsi la tête du serpent brownien peut être vue comme un mouvement brownien indexé par le CRT.

1.2.6 Une nouvelle mesure d'excursion pour le serpent brownien

Dans la seconde partie de cette thèse, on s'est intéressé aux excursions du serpent brownien, et on a développé une théorie des excursions pour le serpent brownien, qui a de nombreux points communs avec la théorie d'Itô des excursions pour le mouvement brownien. On présente ici les résultats pour les excursions en dehors de 0, par analogie avec les excursions du mouvement brownien. Pour des raisons techniques, nous avons majoritairement travaillé avec les excursions au-dessus du minimum, qui seront mentionnées dans la suite.

On regarde les composantes connexes de l'ensemble ouvert

$$\{u \in \mathcal{T}_\zeta : V_u \neq 0\}$$

des sommets de l'arbre codé par ζ dont l'étiquette est non nulle. On note ces composantes connexes $(\mathcal{C}_i)_{i \in I}$. Pour $i \in I$, le processus $(V_u)_{u \in \mathcal{C}_i}$ des étiquettes sur les sommets est encore un processus aléatoire indexé par un arbre aléatoire. C'est l'excursion appelée E_i.

Le résultat principal obtenu décrit la loi des excursions $(E_i)_{i\in I}$. On va introduire une mesure d'excursion $\mathbb{M}_0$ dont le rôle est similaire à celui de la mesure d'Itô d'excursions. Dans un premier temps on expose des notations qui permettent de donner une définition précise des excursions E_i.

Chacune des composantes connexes $\mathcal{C}_i$ est bijectivement associée à un sommet u_i de $\mathcal{T}_\zeta$ qui vérifie

— $V_u = 0$;

— u a un descendant strict v tel que les étiquettes le long de la géodésique reliant u et v ne s'annulent pas (à part en u).

On montre que u_i n'est pas un point de branchement de $\mathcal{T}_\zeta$ et donc il existe deux instants bien définis $0 < a_i < b_i < \sigma$ tels que

$$p_\zeta(a_i) = p_\zeta(b_i) = u_i.$$

On définit un élément $W^{(u_i)}$ de $C(\mathbb{R}_+, \mathbb{R})$ en posant pour $s \geq 0$

$$\zeta_s^{(u_i)} = \zeta_{(a_i+s)\wedge b_i} - \zeta_{a_i}$$

puis

$$W_s^{(u_i)}(t) = W_{(a_i+s)\wedge b_i}(\zeta_{a_i} + t)$$

pour $0 \leq t \leq \zeta_s^{(u_i)}$. Remarquons que les notations utilisées ici sont analogues à celles de la partie 1.2.4.

Nous considérons ensuite

$$\tilde{W}_s^{(u_i)} = W^{(u_i)}_{\eta_s^{(u_i)}}$$

où $\eta_s^{(u_i)} = \inf\{r \geq 0 : \int_0^r dt \mathbf{1}_{\{\zeta_t^{(u_i)} \leq \tau_0^*(W_t^{(u_i)})\}} > s\}$, avec $\tau_0^*(w) = \inf\{t > 0 : w(t) = 0\}$. De cette manière, on garde seulement les trajectoires après l'instant ζ_{a_i} des descendants de u_i qui appartiennent à la composante connexe $\mathcal{C}_i$. Le processus $(\tilde{W}_s^{(u_i)})_{s\geq 0}$ ainsi défini est une représentation mathématique rigoureuse de l'excursion E_i.

Pour $i \in I$, on note ℓ_i le temps local en 0 de la trajectoire W_{a_i}.

Théorème C (Theorem 3.1.1)**.**

Il existe une mesure σ-finie sur $C(\mathbb{R}_+, \mathcal{W}_0)$, appelée $\mathbb{M}_0$, telle que, pour toute fonction Φ mesurable positive sur $\mathbb{R}_+ \times C(\mathbb{R}_+, \mathcal{W}_0)$, on a l'égalité suivante.

$$\mathbb{N}_0\left(\sum_{i\in I} \Phi(\ell_i, W^{(u_i)})\right) = \int_0^\infty d\ell \; \mathbb{M}_0(\Phi(\ell, .)).$$

Notons ici que la présence du temps local permet d'avoir des quantités non nulles et finies. On peut écrire la mesure $\mathbb{M}_0$ sous la forme

$$\mathbb{M}_0 = \frac{1}{2}(\mathbb{N}_0^* + \check{\mathbb{N}}_0^*)$$

où $\mathbb{N}_0^*$ correspond aux excursions positives et $\check{\mathbb{N}}_0^*$ est l'image de $\mathbb{N}_0^*$ par l'application $\omega \mapsto -\omega$. Dans la suite les résultats présentés se concentrent majoritairement sur $\mathbb{N}_0^*$.

Tout d'abord, on énonce la propriété suivante qui sera très utile pour la preuve du résultat principal. Pour $\delta > 0$, sous $\mathbb{N}_0^*$, l'ensemble des excursions de hauteur supérieure à δ a une mesure finie, plus précisément

$$\mathbb{N}_0^*(\{\omega : \sup\{\hat{W}_s(\omega) : s \geq 0\} > \delta\}) = c_0\delta^{-3}$$

où c_0 est une constante qui peut être calculée explicitement.

Il existe aussi d'autres points de vue pour définir $\mathbb{N}_0^*$. D'une part, on peut voir apparaître $\mathbb{N}_0^*$ comme la limite quand ϵ tend vers 0 de la loi de $\tilde{W}$ sous $\mathbb{N}_\epsilon$, où le processus $\tilde{W}$ est construit à partir de W en supprimant les trajectoires qui passent par 0 strictement avant leur temps de vie, avec un changement de temps analogue au $\eta^{(u_i)}$ qui permet de passer de $W^{(u_i)}$ à $\tilde{W}^{(u_i)}$. Cette méthode est similaire à la propriété (1.3) pour la mesure n_+. D'autre part, en utilisant des techniques de réenracinement, on peut donner une description de $\mathbb{N}_0^*$ analogue à la décomposition de Bismut pour la mesure d'Itô donnée dans le Théorème 1.2.7.

Enfin, définissons la mesure de sortie en dehors de $]0, \infty[$ sous $\mathbb{M}_0$.

Proposition D (Proposition 3.1.2)**.**

On peut choisir une suite $(\epsilon_n)_{n\geq 1}$ de réels positifs qui tendent vers 0 telle que, $\mathbb{M}_0$ p.p. la limite suivante existe

$$Z_0^* = \lim_{n\to\infty} \epsilon_n^{-2} \int_0^\infty \mathbf{1}_{\{0<|\hat{W}_s|<\epsilon_n\}} ds,$$

et définit une variable aléatoire positive. De plus, la limite précédente ne dépend pas du choix de la suite $(\epsilon_n)_{n\geq 1}$.

La quantité $Z_0^*(\tilde{W}^{(u_i)})$ permet de mesurer le nombre de trajectoires qui reviennent en 0 et donc la taille de la frontière de la composante connexe $\mathcal{C}_i$.

1.2.7 Résultat sur les excursions du serpent brownien

On rappelle dans un premier temps la définition d'un CSBP (continuous-state branching process), pour une présentation détaillée, on renvoie à [44, Chapter II]).

Definition 1.2.13. *Le CSBP de mécanisme de branchement ψ est un processus de Markov $(X_t)_{t\geq 0}$ dans $\mathbb{R}_+$ dont les noyaux de transition $P_t(x, dy)$ vérifient*

$$\int P_t(x, dy) \exp(-\lambda y) = \exp(-x u_t(\lambda))$$

pour $\lambda \geq 0$, la fonction $(u_t(\lambda))_{t\geq 0, \lambda\geq 0}$ est l'unique solution positive de l'équation

$$u_t(\lambda) + \int_0^t ds \psi(u_s(\lambda)) = \lambda,$$

et la fonction ψ est de la forme

$$\psi(u) = \alpha u + \beta u^2 + \int \pi(dr)(\exp(-ru) - 1 + ru),$$

où $\alpha \geq 0$, $\beta \geq 0$ et π est une mesure σ-finie sur $]0, \infty[$ telle que $\int \pi(dr)(r \wedge r^2) < \infty$.

On introduit ensuite un processus $\mathcal{X}$ appelé processus de sortie du temps local, qui intervient dans le résultat principal sur la loi des excursions. Pour $t > 0$ fixé, $\mathcal{X}_t$ mesure la quantité de sommets u de $\mathcal{T}_\zeta$ d'étiquette 0 tels que la trajectoire entre la racine ρ et u a accumulé un temps local en 0 égal à t. Plus précisément, $\mathcal{X}_t$ est la masse de la mesure de sortie sous $\mathbb{N}_0$ en dehors de $\mathbb{R} \times [0, t[$ du processus

$$\mathbf{W}_s = (W_s(t), L_t^0(W_s))_{0 \leq t \leq \zeta_s}$$

où $L_t^0(W_s)$ est le temps local au niveau 0 à l'instant t de W_s sous $\mathbb{N}_0$. Le processus $(\mathbf{W}_s)_{s \geq 0}$ est un serpent brownien dont le déplacement spatial n'est pas seulement un mouvement brownien mais le couple constitué par un mouvement brownien et son temps local en 0. Pour une description de tels processus, on renvoie à [44].

La loi du processus $(\mathcal{X}_t)_{t>0}$ peut être déterminée à l'aide de résultats sur la loi des mesures de sortie et du théorème de Lévy qui relie le temps local d'un mouvement brownien et son supremum. Le processus $(\mathcal{X}_t)_{t>0}$ est un processus de Markov dont les transitions sont celles d'un CSBP de mécanisme de branchement $\psi(\lambda) = \sqrt{\frac{8}{3}}\lambda^{3/2}$.

On énonce une proposition préliminaire qui est très utile dans la preuve du résultat principal. Rappelons que, pour $i \in I$, ℓ_i est le temps local en 0 de la trajectoire W_{a_i}.

Proposition E (Proposition 3.1.3)**.**

Pour $i \in I$, les quantités ℓ_i correspondent exactement aux instants de saut du processus $(\mathcal{X}_r)_{r>0}$. De plus, pour $i \in I$, la taille $Z_0^*(\tilde{W}^{(u_i)})$ de la frontière de $\mathcal{C}_i$ est égale à la taille $\Delta\mathcal{X}_{\ell_i}$ du saut de $\mathcal{X}$ à l'instant ℓ_i.

Il est maintenant possible d'énoncer le résultat principal.

Théorème F (Theorem 3.1.4)**.**

Sous $\mathbb{N}_0$, conditionnellement au processus de sortie du temps local $(\mathcal{X}_r)_{r>0}$, les excursions $(\tilde{W}^{(u_i)})_{i\in I}$ sont indépendantes et, pour tout $j \in I$, la loi conditionnelle de $\tilde{W}^{(u_j)}$ est donnée par $\mathbb{M}_0(.|Z_0^* = \Delta\mathcal{X}_{\ell_j})$.

1.2.8 Étapes de la preuve du résultat sur les excursions du serpent brownien

Dans cette partie, on expose les principaux outils et arguments utiles pour aboutir au Théorème F. Tout d'abord, pour des raisons techniques, on travaille avec un espace différent de $\mathcal{W}$, noté $\mathcal{S}$ et appelé espace des trajectoires de serpent, que l'on présente maintenant.

Une trajectoire de serpent issue de $x \in \mathbb{R}$ est une application continue

$$\begin{aligned} \omega : \mathbb{R}_+ &\to \mathcal{W}_x \\ s &\mapsto \omega_s \end{aligned}$$

qui vérifie les deux propriétés suivantes.

(i) $\omega_0 = x$ et la quantité $\sigma(\omega) = \sup\{s \geq 0 : \omega_s \neq x\}$ est finie et s'appelle la durée de ω.

(ii) Pour $0 \leq s \leq s'$,

$$\omega_s(t) = \omega_{s'}(t), \text{ pour } 0 \leq t \leq \min_{s \leq r \leq s'} \zeta_{(\omega_r)}.$$

C'est la propriété de serpent.

L'ensemble des trajectoires de serpent issues de x est noté $\mathcal{S}_x$ et on pose

$$\mathcal{S} = \bigcup_{x \in \mathbb{R}} \mathcal{S}_x.$$

L'ensemble $\mathcal{S}$ muni de la distance

$$d_{\mathcal{S}}(\omega, \omega') = |\sigma(\omega) - \sigma(\omega')| + \sup_{s \geq 0} d_{\mathcal{W}}(W_s(\omega), W_s(\omega'))$$

est un espace polonais. On a bijection entre ω et le couple formé par les deux fonctions $s \mapsto \zeta_s(\omega)$ et $s \mapsto \hat{W}_s(\omega)$, ce qui permet de montrer plus aisément des propriétés de convergence pour les trajectoires de serpent.

Notre démarche pour démontrer les Théorèmes C et F consiste à s'intéresser d'abord aux excursions du serpent brownien au-dessus du minimum. On étudie les composantes connexes de l'ouvert $\{v \in \mathcal{T}_\zeta : V_v > \inf_{w \in [\![\rho, v[\![} V_w\}$. Ces composantes connexes sont en bijection avec les sommets u de $\mathcal{T}_\zeta$ tels que

— u n'est pas une feuille ;

— $V_u = \min\{V_v : v \in [\![\rho, u]\!]\}$;

— u a un descendant strict w tel que $V_v > V_u$ pour tout $v \in]\!]u, w[\![$.

De tels sommets s'appellent des débuts d'excursion, et l'excursion correspondant au sommet u est notée $\tilde{W}^{(u)}$ par analogie avec les $\tilde{W}^{(u_i)}$ précédentes.

Dans ce cadre, l'étude est simplifiée car le temps local n'intervient plus et le processus $(\mathcal{X}_r)_{r>0}$ est remplacé par le processus Z des mesures de sortie. Pour $t > 0$, sous $\mathbb{N}_0$, on a

$$Z_t = \mathcal{Z}^{(-t, \infty)}.$$

Sous $\mathbb{N}_0$, Z a la loi de la “mesure d'excursion” du CSBP de mécanisme de branchement donné par $\psi(\lambda) = \sqrt{\frac{8}{3}}\lambda^{3/2}$.

De plus, comme on considère des excursions au-dessus du minimum, on ne travaille qu'avec des excursions positives. On peut alors obtenir un analogue du Théorème C où le temps local ℓ_i est remplacé par le minimum des trajectoires, et $\mathbb{M}_0$ est remplacée par $\mathbb{N}_0^*$ (Theorem 3.3.7).

Ensuite, on peut déterminer explicitement la loi du couple (Z_0^*, σ) sous $\mathbb{N}_0^*$ puis définir $\mathbb{N}_0^*$ conditionnée par $\{Z_0^* = z\}$.

On peut alors énoncer le théorème analogue au Théorème F pour les excursions au-dessus du minimum. On note $\mathcal{D}_Z$ l'ensemble des sauts de Z. On a une correspondance bijective entre $\mathcal{D}_Z$ et les débuts d'excursion au-dessus du minimum.

Théorème G (Theorem 3.7.7)**.**

Sous $\mathbb{N}_0$, conditionnellement à Z, les excursions au-dessus du minimum sont indépendantes et, pour tout $r \in \mathcal{D}_Z$, la loi conditionnelle de l'excursion associée est donnée par $\mathbb{N}_0^*(.|Z_0^* = \Delta Z_r)$.

Exposons maintenant les grandes lignes de la preuve du Théorème G. D'abord on considère les excursions au-dessus du minimum de hauteur supérieure à δ, dont le nombre est noté N_δ. On appelle $u_1^\delta, \dots, u_{N_\delta}^\delta$ les débuts d'excursion correspondants.

Sous $\mathbb{N}_0$, conditionnellement à $\{N_\delta \geq j\}$, l'excursion $\tilde{W}^{(u_j^\delta)}$ est indépendante de $(\tilde{W}^{(u_1^\delta)}, \dots, \tilde{W}^{(u_{N_\delta}^\delta)})$ et a pour loi $\mathbb{N}_0^*(.|M > \delta)$ (Proposition 3.7.1). On ne peut pas avoir de propriété plus forte sur la loi des excursions $(\tilde{W}^{(u_j^\delta)})_{1 \leq j \leq N_\delta}$, c'est-à-dire les variables aléatoires $\tilde{W}^{(u_1^\delta)}, \dots, \tilde{W}^{(u_j^\delta)}$ ne sont pas i.i.d de loi $\mathbb{N}_0^*(.|M > \delta)$ sous $\mathbb{N}_0(.|N_\delta \geq j)$. Cependant, à l'aide d'une suite auxiliaire $(\bar{W}^{\delta,1}, \bar{W}^{\delta,2}, \dots)$ de variables aléatoires i.i.d de loi $\mathbb{N}_0^*(.|M > \delta)$, on peut construire une suite infinie $(W^{\delta,1}, W^{\delta,2}, \dots)$ de variables aléatoires i.i.d de loi $\mathbb{N}_0^*(.|M > \delta)$ dont les N_δ premiers termes sont les $\tilde{W}^{u_j^\delta}$, pour $1 \leq j \leq N_\delta$ (Lemma 3.7.2).

Ensuite, on s'intéresse aux débuts d'excursion et à leurs liens avec les sauts du processus Z des mesures de sortie. On a les deux résultats suivants, qui sont analogues à la Proposition E pour les excursions au-dessus du minimum.

— Chaque instant de saut du processus Z correspond à une valeur de $-V_u$ pour un début d'excursion u (Proposition 3.7.3).

— De plus, la taille du saut de Z à l'instant $-V_u$ est égale à la mesure de sortie en 0 $Z_0^*(\tilde{W}^{(u)})$ de l'excursion correspondante (Proposition 3.7.4).

Enfin, nous pouvons définir une mesure de Poisson sur l'espace $\mathbb{R}_+ \times \mathcal{S}$ d'intensité $dt \times \mathbb{N}_0^*(d\omega)$. Cette mesure de Poisson, dont la construction est plus délicate que dans le cadre des excursions browniennes, permet de retrouver les excursions de hauteur supérieure à δ et de reconstruire complètement le processus Z des mesures de sortie (Proposition 3.7.5, Lemma 3.7.6).

Pour passer du Théorème G au Théorème F, l'argument clé est le théorème de Lévy suivant (voir [63, Chapter VI, Theorem 2.3]).

Theorem 1.2.14. *Si $(B_t)_{t\geq 0}$ est un mouvement brownien issu de 0 et $(L_t^0(B))_{t\geq 0}$ est son temps local en 0, alors le couple de processus*

$$(B_t - \min\{B_r : 0 \leq r \leq t\}, -\min\{B_r : 0 \leq r \leq t\})_{t\geq 0}$$

a même loi que $(|B_t|, L_t^0(B))_{t\geq 0}$.

A l'aide du Théorème 1.2.14, on construit un processus $V^\bullet$ indexé par $\mathcal{T}_\zeta$ ayant la même loi que $|V|$ dont les excursions en dehors de 0 correspondent exactement aux excursions au-dessus du minimum pour V. Ceci permet de comprendre les valeurs absolues des excursions en dehors de 0 à partir des résultats sur les excursions au-dessus du minimum.

Finalement, on traite le problème du signe des excursions en dehors de 0 en utilisant la propriété qui permet de reconstruire un mouvement brownien issu de 0 à partir de sa valeur absolue.

1.2.9 Lien avec les cartes aléatoires

Même si le lien n'est pas clair au premier abord, les résultats de la deuxième partie de cette thèse sont aussi fortement reliés aux cartes aléatoires étudiées dans la première partie. L'introduction de la mesure $\mathbb{N}_0^*$ a été motivée en partie par les résultats récents de [25] autour du plan brownien, qui est une version infinie de la carte brownienne.

La représentation du plan brownien décrite dans [25] fait intervenir un arbre infini $\mathcal{T}_\infty$ comportant une "épine" isométrique à $[0,\infty[$ et des sous-arbres branchant le long de cette épine qui sont des arbres browniens. Pour définir le plan brownien $\mathcal{P}_\infty$, on écrit $\mathcal{P}_\infty$ comme un quotient de l'arbre $\mathcal{T}_\infty$ pour une relation d'équivalence définie à partir d'étiquettes browniennes sur $\mathcal{T}_\infty$. L'article [25] étudie le processus des "hulls" de rayon r dans $\mathcal{P}_\infty$, le hull $B_r^\bullet$ de rayon r étant par définition la boule de rayon r dans laquelle on a rempli les trous (en termes mathématiques, $B_r^\bullet$ est le complémentaire de la composante connexe non bornée du complémentaire de la boule de rayon r). La formule (16) de [25] permet de voir que tout niveau de début d'une excursion au-dessus du minimum pour le mouvement brownien indexé par l'un des sous-arbres branchant sur l'épine de $\mathcal{T}_\infty$ est un point de discontinuité du processus des hulls. A l'instant r_0, le hull "avale" les points de $\mathcal{T}_\infty$ qui sont dans l'excursion correspondant au minimum de niveau r_0 dans le sous-arbre de $\mathcal{T}_\infty$ considéré. Cela conduit à un saut positif pour le volume du hull et à un saut négatif pour sa frontière.

L'interprétation précédente montre que $\mathbb{N}_0^*$ décrit, en un certain sens, les sauts du processus des hulls pour le plan brownien. Cela est confirmé par la Proposition 3.6.2 qui montre que la "loi" du couple (Z_0^*, σ) sous $\mathbb{N}_0^*$ est étroitement liée à la "loi" des sauts du processus correspondant à la "longueur" de la frontière et au volume du hull de rayon r (voir le Théorème 1.3 de [25] et aussi pour une version discrète le Théorème 2 de [24]).

Toute la discussion précédente et en particulier les liens avec [24] suggèrent aussi que la mesure $\mathbb{N}_0^*$ devrait permettre de décrire la loi de la limite d'échelle d'une grande triangulation ou quadrangulation aléatoire de frontière de longueur fixée, de la même manière que $\mathbb{N}_0$ (conditionnée par $\sigma = 1$) permet de construire la carte brownienne. Remarquons que la limite d'échelle de grandes cartes avec frontière a déjà été étudiée par Bettinelli [10] et Bettinelli et Miermont [12], mais avec une représentation différente pour la limite. L'intérêt de notre représentation serait que les étiquettes sous $\mathbb{N}_0^*$ correspondraient aux distances depuis la frontière.

Pour compléter ce lien avec les cartes aléatoires, remarquons que la deuxième partie de cette thèse est reliée aux travaux récents de Miller et Sheffield [59, 60], dont le but est de montrer l'équivalence entre la carte brownienne et la gravité quantique de Liouville de paramètre $\gamma = \sqrt{8/3}$. Dans [59], les excursions du serpent brownien au-dessus du minimum définies ici sont utilisées pour introduire la notion de disque brownien, correspondant aux "bulles" qui apparaissent quand on découvre la carte brownienne. On renvoie à la définition de μ_{DISK}^L [59, Proposition 4.4]. Un des points-clés dans [59] est l'idée de se servir des disques browniens pour reconstruire la carte brownienne.

Chapitre 2

Rescaled bipartite planar maps converge to the Brownian map

Cette partie correspond à l'article [1], accepté pour publication.

Contenu de ce chapitre

Abstract

For every integer $n \geq 1$, we consider a random planar map $\mathcal{M}_n$ which is uniformly distributed over the class of all rooted bipartite planar maps with n edges. We prove that the vertex set of $\mathcal{M}_n$ equipped with the graph distance rescaled by the factor $(2n)^{-1/4}$ converges in distribution, in the Gromov-Hausdorff sense, to the Brownian map. This complements several recent results giving the convergence of various classes of random planar maps to the Brownian map.

Pour tout entier n strictement positif, on considère une carte planaire aléatoire $\mathcal{M}_n$ de loi uniforme sur l'ensemble des cartes biparties enracinées à n arêtes. On montre que l'ensemble des sommets de $\mathcal{M}_n$ muni de la distance de graphe renormalisée par $(2n)^{-1/4}$ converge en loi au sens de Gromov-Hausdorff vers la carte brownienne. Ce travail complète une série de résultats de convergence de différents modèles de cartes aléatoires vers la carte brownienne.

2.1 Introduction

Much attention has been given recently to the convergence of large random planar maps viewed as metric spaces to the continuous random metric space known as the Brownian map. See in particular [3, 9, 48, 58]. The main goal of the present work is to provide another interesting example of these limit theorems, in the case of bipartite planar maps with a fixed number of edges.

Recall that a planar map is a proper embedding of a finite connected graph in the two-dimensional sphere, viewed up to orientation-preserving homeomorphisms of the sphere. The faces of the map are the connected components of the complement of edges. The degree of a face is the number of edges incident to it, with the convention that, if both sides of an edge are incident to the same face, then this edge is counted twice in the degree of the face. A planar map is rooted if there is a distinguished oriented edge, which is called the root edge.

We consider only bipartite planar maps in the present work. A planar map is bipartite if its vertices can be colored with two colors, in such a way that two vertices that have the same color are not connected by an edge (in particular, there are no loops). This is equivalent to the property that all faces of the map have an even degree.

If M is a planar map, the vertex set of M is denoted by $V(M)$, and the usual graph distance on $V(M)$ is denoted by d_{gr}^{M}. Let $\mathbf{M}_n^b$ stand for the set of all rooted bipartite maps with n edges.

Theorem 2.1.1. *For every $n \geq 1$, let $\mathcal{M}_n$ be uniformly distributed over $\mathbf{M}_n^b$. Then,*

$$\left(V(\mathcal{M}_n), 2^{-1/4} n^{-1/4} d_{\mathrm{gr}}^{\mathcal{M}_n}\right) \xrightarrow[n\to\infty]{(d)} (\mathbf{m}_\infty, D^*)$$

where $(\mathbf{m}_\infty, D^*)$ is the Brownian map. The convergence holds in distribution in the space $(\mathbb{K}, d_{GH})$, where $\mathbb{K}$ is the set of all isometry classes of compact metric spaces and d_{GH} is the Gromov-Hausdorff distance.

A brief presentation of the Brownian map will be given in Section 2.5 below. See [48] and the references therein for more information about this random compact metric space.

As mentioned above, several limit theorems analogous to Theorem 2.1.1 have been proved for other classes of random planar maps. The case of p-angulations, which are planar maps where all faces have the same degree p, has received particular attention. Le Gall [48] proved the convergence in distribution of rescaled p-angulations with a fixed number of faces to the Brownian map, both when $p = 3$ (triangulations) and when $p \geq 4$ is even. The case of quadrangulations ($p = 4$) has been treated independently by Miermont [58]. More recently, similar results have been obtained for random planar maps with local constraints: Beltran and Le Gall [9] proved the convergence to the Brownian map for quadrangulations with no pendant vertices, and Addario-Berry and Albenque [3] discussed similar results for simple triangulations or quadrangulations, where there are no loops or multiple edges.

All these papers however deal with random planar maps conditioned to have a fixed number of faces. In our setting, it would make no sense to consider the uniform distribution over all bipartite planar maps with a given number of faces, since there are infinitely many such planar maps. Similarly it would make no sense to condition on the number of vertices, and for this reason we consider conditioning on the number of edges, which results in certain additional technical difficulties.

In order to prove Theorem 2.1.1, we first establish a similar result for planar maps that are both rooted and pointed (this means that, in addition to the root edge there is a distinguished vertex, which we call the origin of the map).

As in several of the previously mentioned papers, the proof of this result relies on the combinatorial bijections of Bouttier, di Francesco and Guitter [17] between (rooted and pointed) bipartite planar maps and certain labeled two-type plane trees. Let $\mathbf{M}_n^{b\bullet}$ denote the set of all rooted and pointed planar bipartite maps with n edges, and let $\mathcal{M}_n^\bullet$ be uniformly distributed over $\mathbf{M}_n^{b\bullet}$. The random tree associated with $\mathcal{M}_n^\bullet$ via the Bouttier, di Francesco, Guitter bijection is identified as a (labeled) two-type Galton-Watson tree with explicit offspring distributions, conditioned to have a fixed total progeny (see Proposition 2.3.1 below). In order to prove the convergence to the Brownian map, an important technical step is then to derive asymptotics for the contour and label functions associated with this conditioned tree (Theorem 2.4.1). Such asymptotics for conditioned two-type Galton-Watson trees have been discussed in [52] and [57]. However both these papers consider conditioning on the number of vertices of one type, which makes it easier to derive the desired asymptotics from the case of usual (one-type) Galton-Watson trees. The fact that we are here conditioning on the total number of vertices creates a significant additional difficulty, which we handle through an absolute continuity argument similar to the ones used in Section 6 of [41]. A useful technical ingredient is a seemingly new definition of a "modified" Lukasievicz path associated with a two-type tree, which might be of independent interest. This new definition is somehow related to a bijection of Janson and Stefánsson [36] between one-type and two-type trees.

As we were finishing the first version of the present article, we learnt of the very recent paper [11], which obtains a result similar to ours for *general* planar maps. The arguments of [11] might also be applicable to the bipartite case, but the methods seem quite different from the ones that are presented here.

We finally note the simple scaling constant $2^{-1/4}$ in Theorem 2.1.1. As far as we know, this value is different from the ones already computed for other classes of maps. The analogous constant for uniform general maps with n edges [11] is $(9/8)^{1/4}$ and the one for uniform quadrangulations with n edges (n even) [48, 58] is $(9/4)^{1/4}$.

The paper is organized as follows. Section 2 introduces our main notation and definitions, and recalls the key bijection of [17] between rooted and pointed bipartite maps and labeled two-type trees. In Section 3, we identify the distribution of the random two-type tree associated to a map uniformly distributed over $\mathbf{M}_n^{b\bullet}$, and we introduce its "modified" Lukasievicz path. Section 4 is devoted to the asymptotics of the contour and label functions coding the two-type tree. Section 5 gives the proof of the statement analogous to Theorem 2.1.1 for rooted and pointed maps. Finally, Section 6 explains how to derive Theorem 2.1.1 from the latter statement.

2.2 Bipartite planar maps and trees

2.2.1 Trees

We set $\mathbb{N} = \{1, 2, \ldots\}$ and by convention $\mathbb{N}^0 = \{\emptyset\}$. We introduce the set

$$\mathcal{U} = \bigcup_{n=0}^{\infty} \mathbb{N}^n.$$

An element of $\mathcal{U}$ is a sequence $u = (u^1, \ldots, u^n)$ of elements of $\mathbb{N}$, and we set $|u| = n$ so that $|u|$ represents the "generation" of u. If $u = (u^1, \ldots, u^n)$ and $v = (v^1, \ldots, v^m)$ are two elements of $\mathcal{U}$, then $uv = (u^1, \ldots, u^n, v^1, \ldots, v^m)$ is the concatenation of u and v. The mapping $\pi : \mathcal{U} \setminus \{\emptyset\} \to \mathcal{U}$ is defined by $\pi((u^1, \ldots, u^n)) = (u^1, \ldots u^{n-1})$. One says that $\pi(u)$ is the parent of u, or that u is a child of $\pi(u)$. A plane tree T is a finite subset of $\mathcal{U}$ such that

(i) $\emptyset \in T$;

(ii) if $u \in T \setminus \{\emptyset\}$, then $\pi(u) \in T$;

(iii) for every $u \in T$, there exists an integer $k_u(T) \geq 0$ such that, for every $j \in \mathbb{N}$, $uj \in T$ if and only if $1 \leq j \leq k_u(T)$.

In (iii), the number $k_u(T)$ is interpreted as the number of children of u in T. The size of a plane tree T is $|T| = \#T - 1$, which is the number of edges of T. We denote the set of all plane trees by $\mathbf{A}$.

Consider now a plane tree T and $n = |T|$. We introduce the contour sequence $(u_0, u_1, \ldots, u_{2n})$ of T, which is defined by induction as follows : $u_0 = \emptyset$ and for $i \in \{0, \ldots 2n-1\}$, u_{i+1} is either the first child of u_i that has not appeared yet in the sequence $(u_0, \ldots, u_i)$, or the parent of u_i if all the children of u_i already appeared in the sequence $(u_0, \ldots, u_i)$. Note that $u_{2n} = \emptyset$ and that all vertices of T appear in the sequence $(u_0, \ldots, u_{2n})$ (some appear more than once).

The white vertices of a tree T are all vertices u such that $|u|$ is even and similarly the black vertices are all vertices such that $|u|$ is odd. We denote the sets of white and black vertices of T by T^0 and T^1 respectively.

We will be interested in certain two-type Galton-Watson trees, which we briefly describe here. Let (μ_0, μ_1) be a pair of probability distributions on $\mathbb{Z}_+$ with respective (finite) means m_0 and m_1. We only consider pairs such that $\mu_0(1) + \mu_1(1) < 2$ and $m_0 m_1 \neq 0$. We say that (μ_0, μ_1) is subcritical if $m_0 m_1 < 1$ and critical if $m_0 m_1 = 1$. Assume that the pair (μ_0, μ_1) is critical or subcritical. A random tree ξ whose distribution is specified by

$$P(\xi = T) = \prod_{u \in T^0} \mu_0(k_T(u)) \prod_{u \in T^1} \mu_1(k_T(u)) \ , \quad \forall T \in \mathbf{A}$$

is called a two-type Galton-Watson tree with offspring distributions (μ_0, μ_1). Informally, white vertices have children according to the offspring distribution μ_0 and black vertices have children according to μ_1.

We now introduce labeled trees. A labeled tree is a pair $(T, (\ell(u))_{u \in T^0})$ where T is a plane tree and $(\ell(u))_{u \in T^0}$ is a collection of labels assigned to the white vertices of T, which must satisfy the following properties.

(i) For every $u \in T$, $\ell(u) \in \mathbb{Z}$.

(ii) Let $v \in T^1$ and $k = k_v(T)$. Let $v_1 = v1, \ldots, v_k = vk$ be the children of v in T, and set also $v_0 = v_{k+1} = \pi(v)$. Then, for every $i \in \{0, 1, 2, \ldots, k\}$, $\ell(v_{i+1}) \geq \ell(v_i) - 1$.

The number $\ell(u)$ is called the label of u. Property (ii) means that, if v is a black vertex, the labels i and j of two white vertices adjacent to v and consecutive in clockwise order around v satisfy $j \geq i - 1$.

We denote the set of all labeled trees with n edges by $\mathbf{T}_n$.

A labeled tree $(T, (\ell(u))_{u \in T^0})$ can be coded by a pair of functions. Recall that if $|T| = n$, $(u_0, \ldots, u_{2n})$ is the contour sequence of T. Note that u_i is white if i is even and black if i is odd.

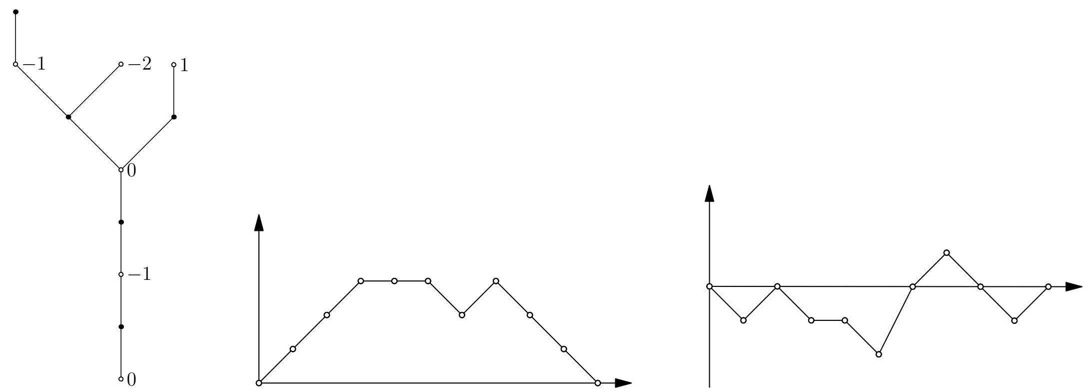

Figure 2.2.1 – A labeled tree T with $n = 10$ edges, the contour function C^{T^0} and the label function L^{T^0}.

We define for $0 \leq i \leq 2n$,

$$C_i^T = |u_i|.$$

We extend C^T to the real interval $[0, 2n]$ by linear interpolation. The function C^T is the contour function of the tree T. For $0 \leq i \leq n$, set $v_i = u_{2i}$. The sequence $(v_0, \ldots, v_n)$ is called the white contour sequence. We then set, for $0 \leq i \leq n$,

$$C_i^{T^0} = \frac{1}{2}|v_i|$$

and

$$L_i^{T^0} = \ell(v_i).$$

We notice that for $0 \leq i \leq n$, we have $C_i^{T^0} = \frac{1}{2}C_{2i}^T$. We also extend both C^{T^0} and L^{T^0} to to the real interval $[0, n]$ by linear interpolation. The function C^{T^0} is called the contour function of T^0 (or the white contour function) and L^{T^0} is called the label function of T^0. See Fig. 2.2.1 for an example. It is easy to verify that the labeled tree $(T, (\ell(u))_{u \in T^0})$ is uniquely determined by the pair (C^T, L^{T^0}) (on the other hand, the pair (C^{T^0}, L^{T^0}) does not give enough information to recover the tree).

2.2.2 The Bouttier-Di Francesco-Guitter bijection

In this section we describe the Bouttier-Di Francesco-Guitter bijection (BDG bijection) between $\mathbf{T}_n \times \{0, 1\}$ and $\mathbf{M}_n^{b\bullet}$. This construction can be found in [17] and in [48] in the particular case of $2p$-angulations.

We start with a labeled tree $(T, (\ell(u))_{u \in T^0}) \in \mathbf{T}_n$ and $\epsilon \in \{0, 1\}$. As above, $(v_0, \ldots, v_n)$ stands for the white contour sequence of T. We suppose that the tree T is represented in the plane in the (obvious) way as suggested by Fig. 2.2.1. A corner of T is a sector around a vertex of T delimited by two consecutive edges in clockwise order. Each corner is given the label of its associated vertex. We note that every $i \in \{0, 1, \ldots, n-1\}$ corresponds to exactly one corner of the vertex v_i (if we move around the tree in clockwise order, the successive white vertices that are visited are $v_0, v_1, \ldots, v_{n-1}, v_n = v_0$ and each visit but the last one corresponds to a new corner), and we will abuse terminology by calling this corner the corner v_i.

We then add an extra vertex ∂ outside the tree T, and we construct a planar map $M^\bullet$, whose vertex set is the union of T^0 and of the extra vertex ∂, as follows: For every $i \in \{0, \ldots, n-1\}$,

— if $\ell(v_i) = \min\{\ell(v), v \in T^0\}$, then we draw an edge of $M^\bullet$ between the corner v_i and ∂;

— if $\ell(v_i) > \min\{\ell(v), v \in T^0\}$, then we draw an edge of $M^\bullet$ between the corner v_i and the corner v_j, where $j = \min\{k > i : \ell(v_k) = \ell(v_i) - 1\}$ if $\{k > i : \ell(v_k) = \ell(v_i) - 1\}$ is nonempty, $j = \min\{k \geq 0 : \ell(v_k) = \ell(v_i) - 1\}$ otherwise.

Thanks to property (ii) of the labels, it is possible to achieve this construction in such a way that edges do not intersect (except at their ends) and do not cross the edges of the tree. The collection of all edges drawn in the preceding construction gives a bipartite planar map $M^\bullet$ with n edges. We then declare that the vertex ∂ is the distinguished vertex of this map and that its root edge is the edge obtained at step $i = 0$ of the preceding construction. The parameter ϵ gives the orientation of this root edge: the root vertex is $\emptyset$ if and only if $\epsilon = 0$. In this way we get a pointed and rooted bipartite planar map $M^\bullet$. See Fig. 2.2.2 for an example with $\epsilon = 0$.

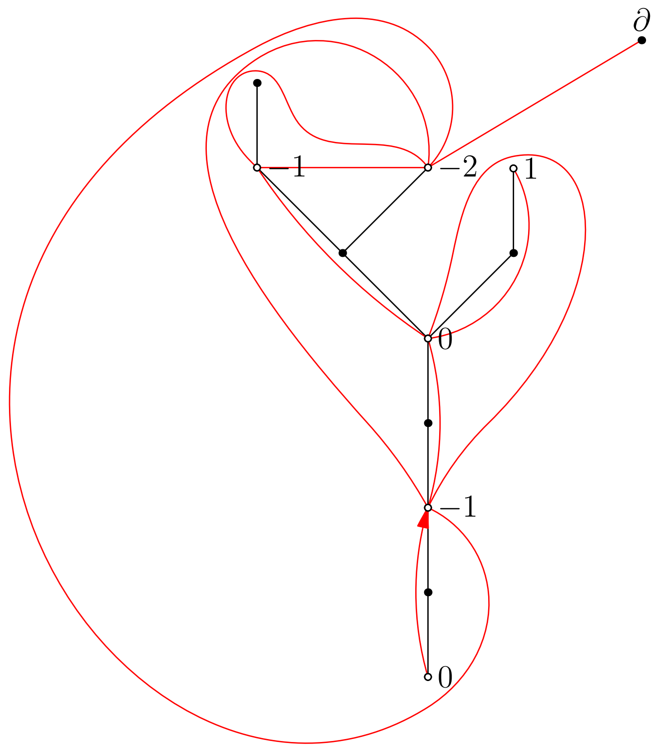

Figure 2.2.2 – The labeled tree T of Fig. 2.2.1 and the associated rooted and pointed bipartite map $M^\bullet$.

The preceding construction yields a bijection from $\mathbf{T}_n \times \{0, 1\}$ onto $\mathbf{M}_n^{b\bullet}$, which is called the Bouttier-Di Francesco-Guitter (BDG) bijection. In this bijection, white vertices of the tree T are identified with vertices of the map $M^\bullet$ other than ∂, and moreover graph distances (in $M^\bullet$) from ∂ are related to labels on T by the formula

$$d_{\mathrm{gr}}^{M^\bullet}(\partial, u) = \ell(u) - \min\{\ell(v), v \in T^0\} + 1, \tag{2.1}$$

for every $u \in T^0$. There is no such expression for $d_{\mathrm{gr}}^{M^\bullet}(u, v)$ when u and v are arbitrary vertices of

$M^\bullet$, but the following bound will be very useful. Let $i, j \in \{0, \ldots, n\}$ such that $i < j$. Then,

$$d_{\mathrm{gr}}^{M^\bullet}(v_i, v_j) \leq \ell(v_i) + \ell(v_j) - 2\max\{\min\{\ell(v_k), i \leq k \leq j\}, \min\{\ell(v_k), j \leq k \leq i+n\}\} + 2, \quad (2.2)$$

where we made the convention that $v_{n+k} = v_k$ for $0 \leq k \leq n$. The proof of this bound is easily adapted from [47, Lemma 3.1].

2.3 Random trees and their contour functions

2.3.1 The tree associated with a map chosen uniformly in $\mathbf{M}_n^{b\bullet}$.

Let $\mathcal{M}_n^\bullet$ be uniformly distributed over the set $\mathbf{M}_n^{b\bullet}$, as in Section 1. We let $(\mathcal{T}_n, (\ell_n(u))_{u \in \mathcal{T}_n^0})$ be the random labeled tree associated with $\mathcal{M}_n^\bullet$ by the previously described BDG bijection. The next proposition determines the distribution of this random tree.

Proposition 2.3.1. *Let (μ_0, μ_1) be the pair of probability measures on $\mathbb{Z}_+$ defined by*

$$\begin{cases} \mu_0(k) = \dfrac{2}{3}\left(\dfrac{1}{3}\right)^k \\ \mu_1(k) = \dfrac{3}{8}\dbinom{2k+1}{k}\left(\dfrac{3}{16}\right)^k \end{cases}$$

for every integer $k \geq 0$. The mean of μ_0 is $1/2$ and the mean of μ_1 is 2, so that the pair (μ_0, μ_1) is critical.

Then the random tree $\mathcal{T}_n$ is a two-type Galton-Watson tree with offspring distributions (μ_0, μ_1) conditioned to have n edges. Furthermore, conditionally given $\mathcal{T}_n$, the labels $(\ell_n(u))_{u \in \mathcal{T}_n^0}$ are uniformly distributed over all admissible labelings.

Proof. Clearly it is enough to determine the law of $\mathcal{T}_n$. We observe that, if T is a plane tree and if u is a black vertex of T with k children, there are $\binom{2k+1}{k}$ possible choices for the increments of labels of white vertices around u. Fix $a \in (0,1)$ and $b \in (0, 1/4)$, and set for every $k \geq 0$,

$$\begin{cases} \nu_0(k) = (1-a)a^k \\ \nu_1(k) = B\dbinom{2k+1}{k}b^k \end{cases}$$

where B is determined by the requirement that ν_1 is a probability measure on $\mathbb{Z}_+$:

$$B = \frac{2b\sqrt{1-4b}}{1-\sqrt{1-4b}}.$$

Assume that (ν_0, ν_1) is subcritical or critical. If θ is a two-type Galton-Watson tree with offspring distributions (ν_0, ν_1), then, for every plane tree T with n edges,

$$P(\theta = T) = \prod_{u \in T^0} \nu_0(k_T(u)) \prod_{u \in T^1} \nu_1(k_T(u)).$$

Writing N_0, respectively N_1, for the number of white, respectively black, vertices of T, we get

$$\begin{aligned} P(\theta = T) &= (1-a)^{N_0} a^{N_1} B^{N_1} b^{N_0 - 1} \prod_{u \in T^1} \binom{2k_T(u)+1}{k_T(u)} \\ &= \frac{1}{b}((1-a)b)^{N_0}(aB)^{N_1} \prod_{u \in T^1} \binom{2k_T(u)+1}{k_T(u)}. \end{aligned}$$

On the other hand, the quantity $P(\mathcal{T}_n = T)$ is proportional to the number of possible labelings of T, so that

$$P(\mathcal{T}_n = T) = c_n \prod_{u \in T^1} \binom{2k_T(u)+1}{k_T(u)},$$

where c_n is the appropriate normalizing constant. If a and b are such that

$$(1-a)b = aB, \tag{2.3}$$

noting that $N_0 + N_1 = n+1$, we see that $P(\mathcal{T}_n = T)$ coincides with $P(\theta = T)$ up to a multiplicative constant that depends only on n, and it follows that

$$P(\mathcal{T}_n = T) = P(\theta = T \,\big|\, |\theta| = n). \tag{2.4}$$

The condition (2.3) holds if

$$a = \frac{1}{3}, \; b = \frac{3}{16}.$$

Furthermore, for these values of a and b, we can verify that the mean of ν_0 is $1/2$ and the mean of ν_1 is 2, so that the pair (ν_0, ν_1) is critical. It then follows from the preceding considerations and in particular from (2.4) that the law of $\mathcal{T}_n$ is as stated in the proposition. $\square$

Remark 2.3.2. *One can easily compute the respective variances σ_0^2 and σ_1^2 of the probability measures μ_0 and μ_1. For future reference, we record that*

$$\sigma_0^2 = \frac{3}{4} \; , \; \sigma_1^2 = \frac{15}{2}.$$

2.3.2 The white contour function and an associated random walk

Consider a random labeled tree $(\mathcal{T}, (\ell(u))_{u \in \mathcal{T}})$, such that $\mathcal{T}$ is a two-type Galton-Watson tree with offspring distributions (μ_0, μ_1) given by Proposition 2.3.1, and conditionally on $\mathcal{T}$ the labels $(\ell(u))_{u \in \mathcal{T}}$ are uniformly distributed among admissible labelings. Let N denote the (random) number of edges of $\mathcal{T}$, and write $(u_0, \ldots, u_{2N})$ for the contour sequence of $\mathcal{T}$.

For every integer $k \geq 0$, we let the σ-field $\mathcal{F}_k$ be generated by the following random variables:

- the quantity $k \wedge N$ and the vertices $u_0, u_1, \ldots, u_{2(k \wedge N)}$ of $\mathcal{T}$;
- the labels $\ell(u_0), \ell(u_2), \ldots, \ell(u_{2(k \wedge N)})$ of the white vertices $u_0, u_2, \ldots, u_{2(k \wedge N)}$;
- for every odd integer i such that $0 < i < 2(k \wedge N)$, the quantity $k_{\mathcal{T}}(u_i)$ and the labels $\ell(u_i j)$, $1 \leq j \leq k_{\mathcal{T}}(u_i)$, of the (white) children of the black vertex u_i.

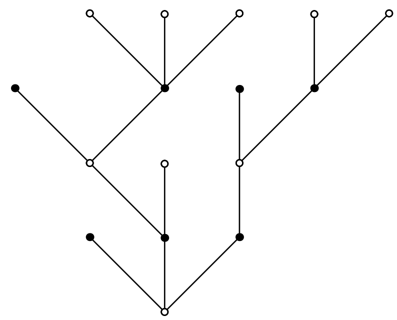

Figure 2.3.1 – A realization of the tree $\mathcal{T}$.

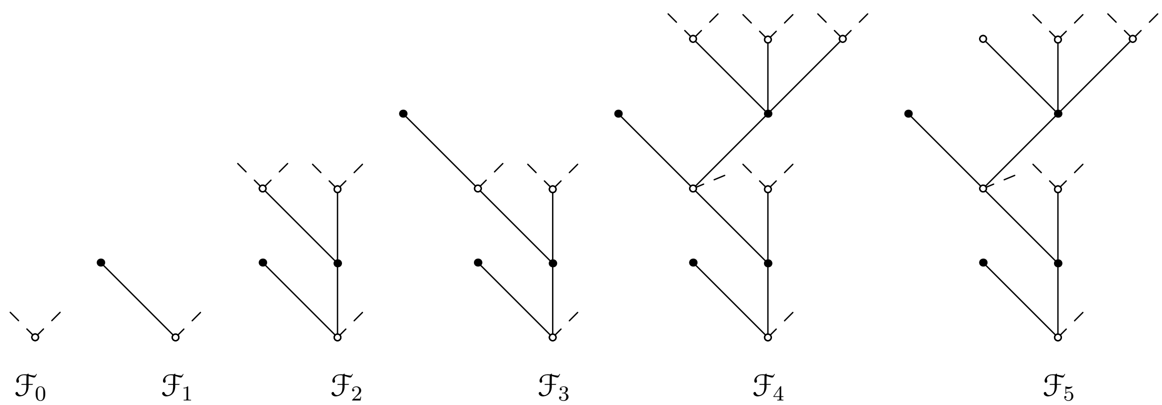

Figure 2.3.2 – The information about the tree $\mathcal{T}$ of Fig. 2.3.1 given by the σ-field $\mathcal{F}_k$ for $k = 0, 1, \dots, 5$. The dashed lines correspond to the "active" white vertices. In this example, $Y_0 = Y_1 = 1$, $Y_2 = Y_3 = 3$, $Y_4 = 6$, $Y_5 = 5$, etc.

Fig. 2.3.1 below gives a realization of the tree $\mathcal{T}$ and Fig. 2.3.2 shows the information discovered by the σ-field $\mathcal{F}_k$ for $k = 0, 1, \dots, 5$. This information should also include the labels of the white vertices that are successively revealed, but these labels are not shown here.

We also introduce a random sequence $(Y_0, Y_1, \dots Y_{N+1})$, which is defined by induction by setting $Y_0 = 1$ and, for every $0 \le k \le N$:

— if u_{2k} has at least one child that does not appear among $u_0, u_1, \dots, u_{2k-1}$, then $Y_{k+1} = Y_k + k_{\mathcal{T}}(u_{2k+1})$,

— otherwise $Y_{k+1} = Y_k - 1$.

Informally, for $0 \le k \le N$, Y_k counts the number of white vertices that have been visited before time $2k$ by the contour sequence, or are children of black vertices visited before time $2k$, and are still "active" at time $2k$. Saying that a white vertex is still active means that it may have children that have not yet been visited at time $2k$. It is easy to verify that the random variable Y_k (which is only defined on the $\mathcal{F}_k$-measurable set $\{k \le N+1\}$) is $\mathcal{F}_k$-measurable and $Y_k \ge 1$ if $0 \le k \le N$, whereas $Y_{N+1} = 0$.

The white contour function of $\mathcal{T}$ can be expressed in terms of the sequence $(Y_k)_{0 \le k \le N+1}$ via the formula: for $0 \le k \le N$,

$$C_k^{\mathcal{T}^0} = \mathrm{Card}\{j \in \{0, \dots, k-1\} : Y_j < \inf\{Y_l : j+1 \leqslant l \leqslant k\}\}. \tag{2.5}$$

We leave the easy verification of (2.5) to the reader. Note that the sequence $(Y_0, Y_1, \dots, Y_{N+1})$ is a kind of "Lukasiewicz path" for our two-type tree, and that the preceding display is analogous to the formula relating the Lukasiewicz path of a (one-type) tree to its height function, see e.g. [43, Proposition 1.2]. We also notice that the indices j counted in $C_k^{\mathcal{T}^0}$ correspond to white vertices on the lineage path of v_k in the tree $\mathcal{T}$.

For every $k \geq 0$, we denote the indicator function of the event

$$\{k \leq N \text{ and the vertex } u_{2k} \text{ still has a non visited black child at instant } 2k\}$$

by η_k. Then, conditionally on $\mathcal{F}_k$ and on the event $\{k \leq N\}$, η_k is distributed as a Bernoulli random variable with parameter $\frac{1}{3}$. Furthermore, conditionally on $\mathcal{F}_k$ and on the event $\{\eta_k = 1\}$, $k_{\mathcal{T}}(u_{2k+1}) = Y_{k+1} - Y_k$ is distributed according to μ_1. On the other hand, if $k \leq N$ and $\eta_k = 0$, we have $Y_{k+1} = Y_k - 1$.

Let ν be the probability measure on $\{-1, 0, 1, \ldots\}$ defined by

$$\begin{cases} \nu(-1) = \dfrac{2}{3} \\ \nu(k) = \dfrac{1}{3}\mu_1(k) \text{ for } k \geq 0 \end{cases}$$

and let $(S_k)_{k\geq 0}$ be a random walk with jump distribution ν starting from $S_0 = 1$. It follows from the preceding discussion that $(Y_0, Y_1, \ldots, Y_{N+1})$ has the same distribution as $(S_0, S_1, \ldots, S_\tau)$ where $\tau = \inf\{n \geq 0, S_n = 0\}$. The distribution ν is centered and has a finite variance $\sigma^2 = 9/2$.

Remark 2.3.3. *It follows that $N+1$ (which is the total progeny of the two-type tree $\mathcal{T}$) has the same distribution as τ, and it is well known that this distribution is the same as the total progeny of a (one-type) Galton-Watson tree with offspring distribution $\mu(k) = \nu(k-1)$ for every $k \geq 0$. A similar fact would hold for any (critical or subcritical) two-type Galton-Watson tree such that the offspring distribution of white vertices is geometric. This was already observed in the recent article of Janson and Stefánsson [36], with a different approach involving a bijection between one-type and two-type trees: See [22, Proposition 3.6] for a statement derived from [36], which corresponds exactly to the previous discussion.*

In the remaining part of this section, we state a couple of useful facts about the random walk S, which are variants of results than can be found in [43, Lemmas 1.9 to 1.12]. For $m \in \mathbb{Z}_+$, we introduce the "time-reversed" random walk $\hat{S}^m$ defined by

$$\hat{S}_k^m = S_m - S_{m-k} + 1$$

for $0 \leq k \leq m$. The random walk $(\hat{S}_k^m, 0 \leq k \leq m)$ has the same distribution as $(S_k, 0 \leq k \leq m)$. We set

$$M_m = \sup\{S_k, 0 \leq k \leq m\}$$

and

$$I_m = \inf\{S_k, 0 \leq k \leq m\}.$$

For every sequence $\omega = (\omega(0), \omega(1), \ldots)$ of integers of length at least m, we set

$$F_m(\omega) = \mathrm{Card}\{k \in \{1, \ldots m\} : \omega(k) > \sup\{\omega(j), 0 \leq j \leq k-1\}\}.$$

We then define $(R_m)_{m\geq 0}$ and $(K_m)_{m\geq 0}$ by

$$R_m = F_m(\hat{S}^m)\ , \quad K_m = F_m(S).$$

Note that we have

$$R_m = \mathrm{Card}\{j \in \{0,\dots,m-1\} :\ S_j < \inf\{S_l : j+1 \leq l \leq m\}\}. \tag{2.6}$$

(compare with (2.5)).

Lemma 2.3.4. *We define by induction $T_0 = 0$, and for every integer $j \geq 1$,*

$$T_{j+1} = \inf\{k > T_j : S_k > S_{T_j}\}.$$

Then the random variables $(S_{T_j} - S_{T_{j-1}})_{j\geq 1}$ are independent and identically distributed, and the distribution of $S_{T_1} - S_{T_0} = S_{T_1} - 1$ is given by

$$P(S_{T_1} - 1 = k) = \frac{3}{2}\nu([k,\infty))$$

for $k \geq 1$.

Proof. The fact that the random variables $(S_{T_j} - S_{T_{j-1}})_{j\geq 1}$ are i.i.d. is immediate from the strong Markov property. Let S' be a random walk with jump distribution ν, starting from $S'_0 = 0$, and $T'_1 = \inf\{k > 0, S'_k \geq 0\}$. By [43, Lemma 1.9], we have $P(S'_{T'_1} = k) = \nu([k,\infty))$ for every $k \geq 0$. Next it is clear that the law of $S_{T_1} - 1$ coincides with the conditional law of $S'_{T'_1}$ knowing that $\{S'_{T'_1} > 0\}$. The desired result easily follows. □

It follows that the distribution of $S_{T_1} - 1$ has a finite first moment, given by $E(S_{T_1} - 1) = 3\sigma^2/4$. A simple argument using the law of large numbers then shows that

$$\frac{M_m}{K_m} \underset{m\to\infty}{\longrightarrow} \frac{3\sigma^2}{4}$$

almost surely. The next lemma provides estimates for "moderate deviations" in this convergence.

Lemma 2.3.5. *Let $\epsilon \in \left(0, \frac{1}{4}\right)$. We can find $\epsilon' > 0$ and an integer $n_0 \geq 1$ such that for $m \geq n_0$ et $l \in \{0,\dots,m\}$, we have the bound*

$$P\left(\left|M_l - \frac{3\sigma^2}{4}K_l\right| > m^{1/4+\epsilon}\right) < \exp\left(-m^{\epsilon'}\right).$$

Proof. The arguments are easily adapted from the proof of Lemma 1.11 in [43]. □

2.4 Convergence of the contour and the label functions

We keep the notation $(\mathcal{T}, (\ell(u))_{u\in\mathcal{T}})$ for a random labeled tree such that $\mathcal{T}$ is a two-type Galton-Watson tree with offspring distributions (μ_0, μ_1) given by Proposition 2.3.1, and conditionally on $\mathcal{T}$ the labels $(\ell(u))_{u\in\mathcal{T}}$ are uniformly distributed among admissible labelings. As previously, $N = |\mathcal{T}|$. In this section, we discuss the convergence as $n \to \infty$ of the conditional distribution of the pair $(n^{-1/2}C^{\mathcal{T}^0}_{nt}, n^{-1/4}L^{\mathcal{T}^0}_{nt})_{0\le t\le 1}$ knowing that $N = n$ (recall the notation $C^{\mathcal{T}^0}$ and $L^{\mathcal{T}^0}$ for the contour function and the label function of $\mathcal{T}^0$, see the end of subsection 2.2.1). The whole section is devoted to the proof of the next theorem.

Theorem 2.4.1. *The conditional distribution of*

$$\left(\frac{1}{\sqrt{n}}C^{\mathcal{T}^0}_{nt}, \frac{1}{n^{1/4}}L^{\mathcal{T}^0}_{nt}\right)_{0\le t\le 1}$$

knowing that $N = n$ converges as $n \to \infty$ to the law of

$$\left(\frac{4\sqrt{2}}{9}\,\mathbf{e}_t, 2^{1/4}\,Z_t\right)_{0\le t\le 1}$$

where $\mathbf{e}$ is a normalized Brownian excursion and Z is the Brownian snake driven by this excursion.

Remark 2.4.2. *We note that, for every $i \in \{0, \dots N+1\}$, $C^{\mathcal{T}}_{2i} = 2C^{\mathcal{T}^0}_i$ and $|C^{\mathcal{T}}_{2i+1} - C^{\mathcal{T}}_{2i}| = 1$. From this trivial observation, the convergence in distribution of Theorem 2.4.1 also implies that $\left(\frac{1}{2\sqrt{n}}C^{\mathcal{T}}_{2nt}\right)_{0\le t\le 1}$ converges to $\left(\frac{4\sqrt{2}}{9}\mathbf{e}_t\right)_{0\le t\le 1}$, and the latter convergence holds jointly with that of Theorem 2.4.1. This simple remark will be useful later.*

We recall that a normalized Brownian excursion $\mathbf{e}$ is just a Brownian excursion conditioned to have duration 1, and that the distribution of Z can be described by saying that, conditionally on $\mathbf{e}$, $(Z_t)_{0\le t\le 1}$ is a centered Gaussian process with continuous sample paths, with covariance

$$E[Z_s Z_t \mid \mathbf{e}] = \min_{s\wedge t\le r\le s\vee t} \mathbf{e}_r.$$

It will sometimes be convenient to make the convention that $\mathbf{e}_t = Z_t = 0$ for $t > 1$. Later we will consider the Brownian snake driven by other types of Brownian excursion, or by reflected linear Brownian motion. Obviously this is defined by the same conditional distribution as above.

As we already mentioned in the introduction, Theorem 2.4.1 is closely related to analogous statements proved in [52, 57] for multitype Galton-Watson trees. A major difference however is the fact that [52, 57] condition on the number of vertices of one particular type, and not on the total number of vertices in the tree. Apparently the latter conditioning (on the total size of the tree) cannot be handled easily by the methods of [52, 57]. See in particular the remarks in [52, p.1682].

Let us turn to the proof. We will rely on formula (2.5) for $C^{\mathcal{T}^0}$. In connection with this formula, we recall that $(Y_0, Y_1, \dots, Y_{N+1})$ has the same distribution as $(S_0, S_1, \dots, S_\tau)$, where $(S_k)_{k\ge0}$ is a random walk with jump distribution ν starting from 1, and $\tau = \inf\{n \ge 0 : S_n = 0\}$. It will be convenient to use the notation P_j for a probability measure under which the random walk S starts from j. By standard local limit theorems (see e.g. Theorems 2.3.9 and 2.3.10 in [39]), we have

$$\lim_{m\to\infty} \sup_{j\in\mathbb{Z}} \left(1 \vee \frac{|j|^2}{m}\right) \left|\sqrt{m}P_j(S_m = 0) - \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{j^2}{2\sigma^2 m}\right)\right| = 0. \tag{2.7}$$

Here $\sigma^2 = 9/2$ is the variance of the distribution ν. We also recall Kemperman's formula (see e.g. [62, p.122]). Let $m \geq j \geq 1$ be two integers. Then,

$$P_j(\tau = m) = \frac{j}{m} P_j(S_m = 0). \tag{2.8}$$

Since $N+1$ has the same distribution as τ under P_1, by combining Kemperman's formula with (2.7), we immediately get

$$n^{3/2} P(N = n) \underset{n\to\infty}{\longrightarrow} \frac{1}{\sigma\sqrt{2\pi}} \quad \text{and} \quad n^{1/2} P(N \geq n) \underset{n\to\infty}{\longrightarrow} \frac{2}{\sigma\sqrt{2\pi}} \tag{2.9}$$

First step

Let $\delta \in (0,1)$ and let Ψ be a bounded continuous function on the space $\mathcal{C}([0,1],\mathbb{R}^2)$ of all continuous functions from $[0,1]$ into $\mathbb{R}^2$. Recall the definition of the σ-fields $\mathcal{F}_k$. We have

$$\begin{aligned} &E\left[\Psi\left(\left(\frac{1}{\sqrt{n}} C^{\mathcal{T}^0}_{nt}, \frac{1}{n^{1/4}} L^{\mathcal{T}^0}_{nt}\right), 0 \leqslant t \leqslant 1-\delta\right) \mathbf{1}_{\{N=n\}}\right] \\ &= E\left[\Psi\left(\left(\frac{1}{\sqrt{n}} C^{\mathcal{T}^0}_{nt}, \frac{1}{n^{1/4}} L^{\mathcal{T}^0}_{nt}\right), 0 \leqslant t \leqslant 1-\delta\right) \mathbf{1}_{\{N \geq \lceil (1-\delta)n \rceil\}\}} P(N = n \,|\, \mathcal{F}_{\lceil (1-\delta)n\rceil})\right]. \end{aligned} \tag{2.10}$$

We then need to study the term $P(N = n \,|\, \mathcal{F}_{\lceil (1-\delta)n\rceil})$.

We notice that, conditionally on $\{N \geq \lceil (1-\delta)n \rceil\}$ and on the σ-field $\mathcal{F}_{\lceil (1-\delta)n\rceil}$, the sequence $(Y_{\lceil (1-\delta)n\rceil}, Y_{\lceil (1-\delta)n\rceil+1}, \ldots, Y_{N+1})$ has the same distribution as a random walk with jump distribution ν starting from $Y_{\lceil (1-\delta)n\rceil}$ and stopped when it hits 0. Thus, we apply Kemperman's formula (2.8), and we obtain, still on the event $\{N \geq \lceil (1-\delta)n \rceil\}$,

$$P(N = n \,|\, \mathcal{F}_{\lceil (1-\delta)n\rceil}) = P_{Y_{\lceil (1-\delta)n\rceil}}(\tau = n + 1 - \lceil (1-\delta)n\rceil) = \Phi_n(Y_{\lceil (1-\delta)n\rceil}) \tag{2.11}$$

where $\Phi_n(j) = \frac{j}{m_n} P_j(S_{m_n} = 0)$, for $0 \leq j \leq m_n$, and $m_n = n + 1 - \lceil (1-\delta)n \rceil = \lfloor \delta n \rfloor + 1$.

Lemma 2.4.3. *We have*

$$\lim_{n\to\infty} \sqrt{n}\, E\left[\mathbf{1}_{\{N \geq \lceil (1-\delta)n\rceil\}} \left| n\Phi_n(Y_{\lceil (1-\delta)n\rceil}) - f_\delta\left(\frac{Y_{\lceil (1-\delta)n\rceil}}{\sqrt{m_n}}\right)\right|\right] = 0,$$

where for every $x \geq 0$,

$$f_\delta(x) = \frac{x}{\delta\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right).$$

Proof. We use the local limit theorem (2.7) to evaluate $n\Phi_n(j)$. Remark that

$$n\Phi_n(j) = \frac{n}{m_n} j P_j(S_{m_n} = 0)$$

and $n/m_n \longrightarrow 1/\delta$ as $n \to \infty$. It easily follows from (2.7) that

$$\lim_{n\to\infty} \sup_{0 \leq j \leq m_n} \left| j P_j(S_{m_n} = 0) - \frac{1}{\sigma\sqrt{2\pi}} \frac{j}{\sqrt{m_n}} \exp\left(-\frac{j^2}{2\sigma^2 m_n}\right)\right| = 0.$$

Thus we have

$$\lim_{n\to\infty}\sup_{0\le j\le m_n}\left|n\Phi_n(j)-\frac{1}{\sigma\delta\sqrt{2\pi}}\frac{j}{\sqrt{m_n}}\exp\left(-\frac{j^2}{2\sigma^2 m_n}\right)\right|=0.$$

Recalling the definition of f_δ, we have thus obtained

$$\lim_{n\to\infty}\sup_{0\le j\le m_n}\left|n\Phi_n(j)-f_\delta\left(\frac{j}{\sqrt{m_n}}\right)\right|=0, \tag{2.12}$$

and the result of the lemma follows using also (2.9). □

The next step is given by the following lemma.

Lemma 2.4.4. *We have*

$$\sqrt{n}\,E\left[\mathbf{1}_{\{N\ge\lceil(1-\delta)n\rceil\}}\left|f_\delta\left(\frac{Y_{\lceil(1-\delta)n\rceil}}{\sqrt{m_n}}\right)-f_\delta\left(\frac{3\sigma^2}{4}\frac{C^{\mathfrak{T}^0}_{\lceil(1-\delta)n\rceil}}{\sqrt{m_n}}\right)\right|\right]\underset{n\to\infty}{\longrightarrow}0.$$

Proof. From the fact that $(Y_0,\dots Y_{N+1})$ has the same distribution as $(S_0,\dots,S_\tau)$ under P_1, and formula (2.5), we get that the distribution of $(Y_{\lceil(1-\delta)n\rceil},C^{\mathfrak{T}^0}_{\lceil(1-\delta)n\rceil},N)$ conditionally on $\{N\ge\lceil(1-\delta)n\rceil\}$ is the same as the distribution of $(S_{\lceil(1-\delta)n\rceil},R_{\lceil(1-\delta)n\rceil},\tau-1)$ under P_1 conditionally on $\{\tau>\lceil(1-\delta)n\rceil\}$. Thus the left-hand side of (2.4.4) can be written as

$$\sqrt{n}\,E_1\left[\mathbf{1}_{\{\tau>\lceil(1-\delta)n\rceil\}}\left|f_\delta\left(\frac{S_{\lceil(1-\delta)n\rceil}}{\sqrt{m_n}}\right)-f_\delta\left(\frac{3\sigma^2}{4}\frac{R_{\lceil(1-\delta)n\rceil}}{\sqrt{m_n}}\right)\right|\right].$$

By time reversal, the following identity in distribution holds under P_1, for $0\le l\le m$:

$$(S_l-I_l,R_l)\overset{(d)}{=}(M_l-1,K_l).$$

So Lemma 2.3.5 can be rephrased as follows. Let $\epsilon\in(0,1/4)$. We can find $\epsilon'>0$ and $n_0\geqslant 1$ such that for $m\geqslant n_0$ and $l\in\{0,\dots,m\}$, we have

$$P_1\left(\left|\frac{S_l-I_l+1}{\sqrt{m}}-\frac{3\sigma^2}{4}\frac{R_l}{\sqrt{m}}\right|>m^{-1/4+\epsilon}\right)<\exp\left(-m^{\epsilon'}\right). \tag{2.13}$$

Then, since the function f_δ is bounded and Lipschitz, we have

$$\begin{aligned}&\sqrt{n}\,E_1\left[\mathbf{1}_{\{\tau>\lceil(1-\delta)n\rceil\}}\left|f_\delta\left(\frac{S_{\lceil(1-\delta)n\rceil}}{\sqrt{m_n}}\right)-f_\delta\left(\frac{3\sigma^2}{4}\frac{R_{\lceil(1-\delta)n\rceil}}{\sqrt{m_n}}\right)\right|\right]\\&\le\sqrt{n}\,K_\delta\,E_1\left[\mathbf{1}_{\{\tau>\lceil(1-\delta)n\rceil\}}\left(\left|\frac{S_{\lceil(1-\delta)n\rceil}-\frac{3\sigma^2}{4}R_{\lceil(1-\delta)n\rceil}}{\sqrt{m_n}}\right|\wedge 1\right)\right].\end{aligned}$$

where the constant K_δ only depends on δ. It follows that

$$\begin{aligned}&\sqrt{n}\,E_1\left[\mathbf{1}_{\{\tau>\lceil(1-\delta)n\rceil\}}\left|f_\delta\left(\frac{S_{\lceil(1-\delta)n\rceil}}{\sqrt{m_n}}\right)-f_\delta\left(\frac{3\sigma^2}{4}\frac{R_{\lceil(1-\delta)n\rceil}}{\sqrt{m_n}}\right)\right|\right]\\&\le\sqrt{n}\,K_\delta\,\frac{1}{\sqrt{m_n}}n^{1/4+\epsilon}E_1[\mathbf{1}_{\{\tau>\lceil(1-\delta)n\rceil\}}]\\&+\sqrt{n}\,K_\delta\,P_1\left(\left|S_{\lceil(1-\delta)n\rceil}-\frac{3\sigma^2}{4}R_{\lceil(1-\delta)n\rceil}\right|>n^{1/4+\epsilon},\tau>\lceil(1-\delta)n\rceil\right).\end{aligned}$$

The first term in the sum tends to 0 as $n \to \infty$ thanks to (2.9). We then use the fact that $I_{\lceil (1-\delta)n \rceil} = 1$ on the event $\{\tau > \lceil (1-\delta)n \rceil\}$ and the bound (2.13) to see that the second term also tends to 0. We thus get

$$\sqrt{n}\, E_1 \left[\mathbf{1}_{\{\tau > \lceil (1-\delta)n \rceil\}} \left| f_\delta\left(\frac{S_{\lceil (1-\delta)n \rceil}}{\sqrt{m_n}} \right) - f_\delta\left(\frac{3\sigma^2}{4} \frac{R_{\lceil (1-\delta)n \rceil}}{\sqrt{m_n}} \right) \right| \right] \underset{n\to\infty}{\longrightarrow} 0$$

and our claim follows. $\square$

It follows from Lemmas 2.4.3 and 2.4.4 that

$$\lim_{n\to\infty} \sqrt{n}\, E \left[\left| n\Phi_n(Y_{\lceil (1-\delta)n \rceil}) - f_\delta\left(\frac{3\sigma^2}{4} \frac{C^{\mathfrak{T}^0}_{\lceil (1-\delta)n \rceil}}{\sqrt{m_n}} \right) \right| \mathbf{1}_{\{N \geq \lceil (1-\delta)n \rceil\}} \right] = 0. \tag{2.14}$$

From (2.10) and (2.11), we now obtain

$$\begin{aligned} \lim_{n\to\infty} \Bigg| & n^{3/2} E\left[\Psi\left(\left(\frac{1}{\sqrt{n}} C^{\mathfrak{T}^0}_{nt}, \frac{1}{n^{1/4}} L^{\mathfrak{T}^0}_{nt} \right), 0 \leq t \leq 1-\delta \right) \mathbf{1}_{\{N=n\}} \right] \\ & - \sqrt{n}\, E\left[\Psi\left(\left(\frac{1}{\sqrt{n}} C^{\mathfrak{T}^0}_{nt}, \frac{1}{n^{1/4}} L^{\mathfrak{T}^0}_{nt} \right), 0 \leq t \leq 1-\delta \right) f_\delta\left(\frac{3\sigma^2}{4} \frac{C^{\mathfrak{T}^0}_{\lceil (1-\delta)n \rceil}}{\sqrt{m_n}} \right) \mathbf{1}_{\{N \geq \lceil (1-\delta)n \rceil\}} \right] \Bigg| = 0. \end{aligned} \tag{2.15}$$

Second step

In view of (2.15), we now need to get a limit in distribution for the (rescaled) pair $(C^{\mathfrak{T}^0}_{nt}, L^{\mathfrak{T}^0}_{nt})_{0 \leq t \leq 1-\delta}$ conditioned on the event $\{N \geq \lceil (1-\delta)n \rceil\}$. This is the goal of the next lemma, which is essentially a consequence of results found in [57].

Lemma 2.4.5. *Let $a > 0$. The law under $P(.|N \geq an)$ of the process*

$$\left(\left(\frac{1}{\sqrt{n}} C^{\mathfrak{T}^0}_{(nt)\wedge N}, \frac{1}{n^{1/4}} L^{\mathfrak{T}^0}_{(nt)\wedge N} \right), t \geq 0 \right)$$

converges when $n \to \infty$ to the law of

$$\left(\left(\frac{1}{\tilde{\sigma}} \mathbf{e}^{(a)}_t, \Sigma \sqrt{\frac{2}{\tilde{\sigma}}} Z^{(a)}_t \right), t \geq 0 \right)$$

where $\mathbf{e}^{(a)}$ is a Brownian excursion conditioned to have duration greater than a, $Z^{(a)}$ is the Brownian snake driven by this excursion, and the constants are given by

$$\tilde{\sigma} = \frac{9}{4\sqrt{2}}\ , \quad \Sigma = \sqrt{\frac{9}{8}}.$$

Proof. To relate the convergence of the lemma to the results of [57], we first recall the contour function $C^{\mathfrak{T}}$ and introduce a label function $L^{\mathfrak{T}}$ defined as follows. If $(u_0, u_1, \ldots, u_{2N})$ is the contour sequence of $\mathfrak{T}$, we already saw that $C^{\mathfrak{T}}_i = |u_i|$ and we put $L^{\mathfrak{T}}_i = \ell(u_i)$, for every $i \in \{0, 1, \ldots, 2N\}$, where by convention we have assigned to each black vertex the label of its parent. We then interpolate linearly to define $C^{\mathcal{T}}_t$ and $L^{\mathcal{T}}_t$ for every real $t \in [0,2N]$. It is then enough to verify that the convergence of the lemma holds when $(n^{-1/2}C^{\mathcal{T}^0}_{(nt)\wedge N}, n^{-1/4}L^{\mathcal{T}^0}_{(nt)\wedge N})_{t\geq 0}$ is replaced by $(2^{-1}n^{-1/2}C^{\mathcal{T}}_{(2nt)\wedge(2N)}, n^{-1/4}L^{\mathcal{T}}_{(2nt)\wedge(2N)})_{t\geq 0}$ (see Remark 2.4.2).

We also introduce the variant of the contour function called the height function, and the corresponding variant of the label function. The height function of $\mathcal{T}$ is defined by setting $H^{\mathcal{T}}_i = |w_i|$ for $0 \leq i \leq N$, where $w_0, w_1, \ldots, w_N$ are the vertices of $\mathcal{T}$ listed in lexicographical order, and the modified label function is defined by $\tilde{L}^{\mathcal{T}}_i = \ell(w_i)$ (again we assign to each black vertex the label of its parent). By convention we set $H^{\mathcal{T}}_{N+1} = 0$ and $\tilde{L}^{\mathcal{T}}_{N+1} = 0$. Both $H^{\mathcal{T}}$ and $\tilde{L}^{\mathcal{T}}$ are interpolated linearly to give processes indexed by $[0, N+1]$. Then we may replace $(2^{-1}n^{-1/2}C^{\mathcal{T}}_{(2nt)\wedge(2N)}, n^{-1/4}L^{\mathcal{T}}_{(2nt)\wedge(2N)})_{t\geq 0}$ by $(2^{-1}n^{-1/2}H^{\mathcal{T}}_{(nt)\wedge(N+1)}, n^{-1/4}\tilde{L}^{\mathcal{T}}_{(nt)\wedge(N+1)})_{t\geq 0}$. Indeed it is well known that asymptotics for the height functions, of the type of the convergence (2.16), imply similar asymptotics for the contour functions (and similarly for the label functions) modulo an extra multiplicative factor 2 in the time scaling. See e.g. Section 1.6 in [43] for a precise justification in a slightly different setting. In the case of Galton-Watson trees with a fixed size, the fact that the height process and the contour function converge jointly to the same Brownian excursion is due to Marckert and Mokkadem in [55].

Consider then a sequence $(\mathcal{T}_{(k)}, (\ell_{(k)}(u))_{u\in\mathcal{T}^0_{(k)}})_{k\geq 1}$ of independent labeled trees distributed as $(\mathcal{T}, (\ell(u))_{u\in\mathcal{T}^0})$. Set $N_{(k)} = |\mathcal{T}_{(k)}|$ for every $k \geq 1$. Define the height function H^∞, respectively the label function $\tilde{L}^\infty$, by concatenating the height functions $(H^{\mathcal{T}_{(k)}}_t)_{0\leq t\leq N_{(k)}+1}$, resp. the label functions $(\tilde{L}^{\mathcal{T}_{(k)}}_t)_{0\leq t\leq N_{(k)}+1}$. Then a very special case of Theorems 1 and 3 in [57] gives the convergence in distribution

$$\left(\left(\frac{1}{\sqrt{n}}H^\infty_{nt}, \frac{1}{n^{1/4}}\tilde{L}^\infty_{nt}\right), t\geq 0\right) \xrightarrow[n\to\infty]{(d)} \left(\left(\frac{2}{\tilde{\sigma}}\beta_t, \Sigma\sqrt{\frac{2}{\tilde{\sigma}}}W_t\right), t\geq 0\right) \tag{2.16}$$

where β is a standard reflected linear Brownian motion, and W is the Brownian snake driven by β. Furthermore, the constants $\tilde{\sigma}$ and Σ are as in the statement of the lemma.

Let us comment on the numerical values of the constants $\tilde{\sigma}$ and Σ. Both these constants can be calculated using the formulas found in [57]. More precisely, $\tilde{\sigma}$ is evaluated from formula (2) in [57], using also the numerical values $\sigma_0^2 = 3/4$ and $\sigma_1^2 = 15/2$ for the respective variances of μ_0 and μ_1. Similarly, Σ is computed from the formula in [57, Theorem 3]. When applying this formula, we need to calculate the variance of the difference between the label of the i-th child of a black vertex and the label of the parent of this black vertex, conditionally on the event that the black vertex in consideration has p children (with of course $p \geq i$). This variance is equal to $2i(p-i+1)/(p+2)$, by a calculation found on page 1664 of [52]. The remaining part of the calculation is straightforward, and we leave the details to the reader.

Finally we observe that if $K = \min\{k \geq 1 : N_{(k)} \geq an\}$, the law of the labeled tree $(\mathcal{T}_{(K)}, (\ell_{(K)}(u))_{u\in\mathcal{T}^0_{(K)}})$ is the same as the conditional law $(\mathcal{T}, (\ell(u))_{u\in\mathcal{T}^0})$ knowing that $N \geq an$. On the other hand, the process $(n^{-1/2}H^{\mathcal{T}_{(K)}}_{nt})_{0\leq t\leq n^{-1}(N_{(K)}+1)}$ corresponds to the first excursion of $(n^{-1/2}H^\infty_{nt})_{t\geq 0}$ away from 0 with length greater than or equal to $a+n^{-1}$. By arguments very similar to [43, Proof of Corollary 1.13], we deduce from (2.16) that $(n^{-1/2}H^{\mathcal{T}_{(K)}}_{(nt)\wedge(N_{(K)}+1)})_{t\geq 0}$ converges in distribution to the first excursion of $(\frac{2}{\tilde{\sigma}}\beta_t)_{t\geq 0}$ away from 0 with duration greater than a. This gives the convergence of the first component in Lemma 2.4.5. The convergence of the second component (and the fact that it holds jointly with the first one) is obtained by the same argument. $\square$

By (2.9), we have

$$n^{3/2}P(N=n)\underset{n\to\infty}{\longrightarrow}\frac{1}{\sigma\sqrt{2\pi}}\;,\quad \sqrt{n}\,P(N\geq\lceil(1-\delta)n\rceil)\underset{n\to\infty}{\longrightarrow}\frac{2}{\sigma\sqrt{2\pi}}\,(1-\delta)^{-1/2}.$$

From (2.15) and Lemma 2.4.5, we now get

$$\begin{aligned}&\lim_{n\to\infty}E\left[\Psi\left(\left(\frac{1}{\sqrt{n}}C^{\mathcal{T}^0}_{nt},\frac{1}{n^{1/4}}L^{\mathcal{T}^0}_{nt}\right),0\leq t\leq 1-\delta\right)\,\middle|\,N=n\right]\\&=2(1-\delta)^{-1/2}E\left[\Psi\left(\left(\frac{1}{\tilde\sigma}\mathbf{e}^{(1-\delta)}_t,\Sigma\sqrt{\frac{2}{\tilde\sigma}}Z^{(1-\delta)}_t\right),0\leq t\leq 1-\delta\right)f_\delta\left(\frac{3}{\sqrt{2\delta}}\mathbf{e}^{(1-\delta)}_{1-\delta}\right)\right]\\&=E\left[\Psi\left(\left(\frac{1}{\tilde\sigma}\mathbf{e}^{(1-\delta)}_t,\Sigma\sqrt{\frac{2}{\tilde\sigma}}Z^{(1-\delta)}_t\right),0\leq t\leq 1-\delta\right)g_\delta\left(\mathbf{e}^{(1-\delta)}_{1-\delta}\right)\right],\end{aligned}$$

where, for every $x\geq 0$,

$$g_\delta(x)=2(1-\delta)^{-1/2}\,f_\delta\left(\frac{3}{\sqrt{2\delta}}\,x\right).$$

Recalling the definition of f_δ, and the fact that $\sigma^2=9/2$, we obtain

$$g_\delta(x)=\frac{2x}{\sqrt{2\pi\delta^3(1-\delta)}}\exp\left(-\frac{x^2}{2\delta}\right).$$

It is well known (see formula (1) in [41]) that the function $\omega\longrightarrow g_\delta(\omega(1-\delta))$ is the density (on the space $\mathcal{C}(\mathbb{R}_+,\mathbb{R}_+)$) of the law of the normalized Brownian excursion with respect to the law of the Brownian excursion conditioned to have length greater than $1-\delta$, on the σ-field generated by the coordinates up to time $1-\delta$. Hence we conclude that we have also

$$\begin{aligned}&\lim_{n\to\infty}E\left[\Psi\left(\left(\frac{1}{\sqrt{n}}C^{\mathcal{T}^0}_{nt},\frac{1}{n^{1/4}}L^{\mathcal{T}^0}_{nt}\right),0\leq t\leq 1-\delta\right)\,\middle|\,N=n\right]\\&=E\left[\Psi\left(\left(\frac{1}{\tilde\sigma}\mathbf{e}_t,\Sigma\sqrt{\frac{2}{\tilde\sigma}}Z_t\right),0\leq t\leq 1-\delta\right)\right],\end{aligned}\tag{2.17}$$

where $\mathbf{e}$ and Z are as in the statement of Theorem 2.4.1. Since this holds for every $\delta\in(0,1)$ and since we have $C^{\mathcal{T}^0}_n=L^{\mathcal{T}^0}_n=0$ on the event $\{N=n\}$, we have obtained the convergence of finite-marginal distributions in the convergence of Theorem 2.4.1 (note that $\frac{1}{\tilde\sigma}=\frac{4\sqrt2}{9}$ and $\Sigma\sqrt{\frac{2}{\tilde\sigma}}=2^{1/4}$).

To complete the proof, we still need a tightness argument. But tightness holds if we restrict our processes to $[0,1-\delta]$ by (2.17), and we can then use a time-reversal argument. Indeed $(C^{\mathcal{T}^0}_0,C^{\mathcal{T}^0}_1,\ldots,C^{\mathcal{T}^0}_n)$ and $(C^{\mathcal{T}^0}_n,C^{\mathcal{T}^0}_{n-1},\ldots,C^{\mathcal{T}^0}_0)$ have the same distribution under $P(\cdot\,|\,N=n)$. The similar property does not hold for the label process, but $(L^{\mathcal{T}^0}_n,L^{\mathcal{T}^0}_{n-1},\ldots,L^{\mathcal{T}^0}_0)$ corresponds to the label process for a (conditioned) tree where labels would be generated by using the counterclockwise order instead of the clockwise order, in the constraints of the definition of a labeled tree in subsection 2.1. Clearly, our arguments would go through with this different convention, and so we get the desired tightness also for the label process. This completes the proof of Theorem 2.4.1.

Remark 2.4.6. *The difficulty in proving Theorem 2.4.1 comes from the convergence of labels. If we had been interested only in the convergence of the rescaled contour functions* $\left(\frac{1}{\sqrt{n}}C^{\mathcal{T}^0}_{nt}\right)$, *we could have used formula (2.5) more directly, following the ideas of Marckert and Mokkadem [55]. See also [43, Chapter 1].*

2.5 Convergence towards the Brownian map for rooted and pointed maps

Recall that $\mathcal{M}_n^\bullet$ is a random bipartite planar map uniformly distributed over the set $\mathbf{M}_n^{b\bullet}$ of all bipartite planar rooted and pointed maps with n edges. In this section, we prove the analog of Theorem 2.1.1 when $\mathcal{M}_n$ is replaced by $\mathcal{M}_n^\bullet$, namely

$$\left(V(\mathcal{M}_n^\bullet), 2^{-1/4}n^{-1/4}d_{\mathrm{gr}}^{\mathcal{M}_n^\bullet}\right) \xrightarrow[n\to\infty]{(d)} (\mathbf{m}_\infty, D^*) \tag{2.18}$$

where $(\mathbf{m}_\infty, D^*)$ is the Brownian map.

2.5.1 Definition of the Brownian map

We define the Brownian map following [48, Sect.2.4]. We first need to introduce the CRT (Continuous Real Tree). Let $(\mathbf{e}_s)_{0\le s\le 1}$ be a normalized Brownian excursion. For $s,t\in[0,1]$, we set

$$d_{\mathbf{e}}(s,t) = \mathbf{e}_s + \mathbf{e}_t - 2\min\{\mathbf{e}_r : s\wedge t \le r \le s\vee t\}.$$

We notice that $d_{\mathbf{e}}$ is a random pseudo-metric on $[0,1]$. Consider the equivalence relation defined for $s,t\in[0,1]$ by

$$s \sim_{\mathbf{e}} t \text{ iff } d_{\mathbf{e}}(s,t) = 0.$$

The CRT is then the quotient space $\mathcal{T}_{\mathbf{e}} = [0,1]/\sim_{\mathbf{e}}$, which is equipped with the distance induced by $d_{\mathbf{e}}$. We denote the canonical projection $[0,1]\to\mathcal{T}_{\mathbf{e}}$ by $p_{\mathbf{e}}$.

We then let $Z=(Z_s)_{0\le s\le 1}$ be the Brownian snake driven by $\mathbf{e}$, as in Theorem 2.4.1. We note that $Z_0=0$ and $E((Z_s-Z_t)^2|\mathbf{e}) = d_{\mathbf{e}}(s,t)$. From the last relation, one obtains that $Z_s = Z_t$ for every $s,t\in[0,1]$ such that $d_{\mathbf{e}}(s,t)=0$, a.s. Thus the process Z can be viewed as indexed by the CRT $\mathcal{T}_{\mathbf{e}}$, in such a way that $Z_s = Z_{p_{\mathbf{e}(s)}}$ for $s\in[0,1]$. In the sequel, we will use the notation $Z_s = Z_a$ if $s\in[0,1]$ and $a = p_{\mathbf{e}}(s)$. Using similar techniques as in the proof of the Kolmogorov regularity theorem, one can show that the mapping $a\mapsto Z_a$ is Hölder continuous with exponent $\frac{1}{2}-\epsilon$ with respect to $d_{\mathbf{e}}$, for every $\epsilon\in]0,\frac{1}{2}[$. The pair $(\mathcal{T}_{\mathbf{e}}, (Z_a)_{a\in\mathcal{T}_{\mathbf{e}}})$ is then a continuous analog of discrete labeled trees.

We can now define the Brownian map, as a quotient space of the CRT. For $s,t\in[0,1]$ such that $s\le t$, we set

$$D^0(s,t) = D^0(t,s) = Z_s + Z_t - 2\max\left(\min\{Z_r, r\in[s,t]\}, \min\{Z_r, r\in[0,s]\cup[t,1]\}\right)$$

and for $a,b\in\mathcal{T}_{\mathbf{e}}$,

$$D^0(a,b) = \min\{D^0(s,t) : (s,t)\in[0,1]^2, p_{\mathbf{e}}(s)=a, p_{\mathbf{e}}(t)=b\}.$$

Finally, for $a, b \in \mathcal{T}_{\mathbf{e}}$, let

$$D^*(a,b) = \inf\left\{\sum_{i=1}^{k} D^0(a_{i-1}, a_i)\right\}$$

where the infimum is over all choices of the integer $k \geq 1$ and of the finite sequence $(a_0, \dots a_k)$ of elements of $\mathcal{T}_{\mathbf{e}}$ such that $a_0 = a$ and $a_k = b$. Then, D^* is a pseudo-metric on the CRT $\mathcal{T}_{\mathbf{e}}$, which satisfies $D^* \leq D^0$. One can also interpret D^* as a function on $[0,1]^2$ by setting $D^*(s,t) = D^*(p_{\mathbf{e}}(s), p_{\mathbf{e}}(t))$ for $(s,t) \in [0,1]^2$. Let $\simeq$ be the equivalence relation on $\mathcal{T}_{\mathbf{e}}$ given by

$$a \simeq b \text{ iff } D^*(a,b) = 0.$$

We set

$$\mathbf{m}_\infty = \mathcal{T}_{\mathbf{e}}/\simeq$$

and let $\Pi : \mathcal{T}_{\mathbf{e}} \to \mathbf{m}_\infty$ be the canonical projection. The Brownian map is the space $\mathbf{m}_\infty$ equipped with the distance induced by D^*.

2.5.2 Proof of the convergence towards the Brownian map

As previously, we let $(\mathcal{T}_n, (\ell_n(v))_{v \in \mathcal{T}_n^0})$ be the random labeled tree associated with $\mathcal{M}_n^\bullet$ via the BDG bijection. Recall that $\mathcal{T}_n$ is a two-type Galton-Watson tree with offspring distributions μ_0 and μ_1, conditioned to have n edges. We use the notation $(v_0^n, \dots, v_n^n)$ for the white contour sequence of $\mathcal{T}_n$. Recall that the white vertices in $\mathcal{T}_n$ are identified to vertices of the map $\mathcal{M}_n^\bullet$. For $(i,j) \in \{0, \dots n\}^2$, we set

$$d_n(i,j) = d_{\mathrm{gr}}^{\mathcal{M}_n^\bullet}(v_i^n, v_j^n).$$

We then extend this definition to noninteger values of i and j by putting for $s, t \in [0,n]^2$

$$\begin{aligned} d_n(s,t) =&(s - \lfloor s \rfloor)(t - \lfloor t \rfloor) d_n(\lceil s \rceil, \lceil t \rceil) + (s - \lfloor s \rfloor)(\lceil t \rceil - t) d_n(\lceil s \rceil, \lfloor t \rfloor) \\ &+ (\lceil s \rceil - s)(t - \lfloor t \rfloor) d_n(\lfloor s \rfloor, \lceil t \rceil) + (\lceil s \rceil - s)(\lceil t \rceil - t) d_n(\lfloor s \rfloor, \lfloor t \rfloor). \end{aligned}$$

Recall our convention $v_{n+i}^n = v_i^n$ for $0 \leq i \leq n$. From the bound (2.2), we have for $0 \leq i < j \leq n$,

$$\begin{aligned} d_n(i,j) &\leq \ell_n(v_i^n) + \ell_n(v_j^n) - 2\max\{\min\{\ell_n(v_k^n), i \leq k \leq j\}, \min\{\ell_n(v_k^n), j \leq k \leq i+n\}\} + 2 \qquad (2.19) \\ &= L_i^{\mathcal{T}_n^0} + L_j^{\mathcal{T}_n^0} - 2\max\{\min\{L_k^{\mathcal{T}_n^0}, k \in [i,j]\}, \min\{L_k^{\mathcal{T}_n^0}, k \in [j,n] \cup [0,i]\}\} \end{aligned}$$

From the last bound and the convergence in distribution of the sequence of processes $(n^{-1/4} L_{nt}^{\mathcal{T}_n^0})_{0 \leq t \leq 1}$ (Theorem 2.4.1), one gets that the sequence of the distributions of the processes

$$\left(n^{-1/4} d_n(ns, nt), 0 \leq s, t \leq 1\right)$$

is tight. Using Theorem 2.4.1 and Remark 2.4.2, we see that we can find a sequence $(n_k)_{k \geq 1}$ tending to infinity and a continuous random process $(D(s,t))_{0 \leq s,t \leq 1}$ such that, along $(n_k)_{k \geq 1}$, the following joint convergence in distribution in $\mathcal{C}([0,1]^2, \mathbb{R}^3)$ holds:

$$\left(\frac{9}{8\sqrt{2}} \frac{C_{2nt}^{\mathcal{T}_n}}{n^{1/2}}, 2^{-1/4} \frac{L_{nt}^{\mathcal{T}_n^0}}{n^{1/4}}, 2^{-1/4} \frac{d_n(s,t)}{n^{1/4}}\right)_{0 \leq s,t \leq 1} \xrightarrow[n \to \infty]{} (\mathbf{e}_t, Z_t, D(s,t))_{0 \leq s,t \leq 1}. \qquad (2.20)$$

Using the Skorokhod representation theorem (and recalling that $(\mathcal{T}_n,(\ell_n(v))_{v\in\mathcal{T}_n^0})$ is determined by the pair $(C^{\mathcal{T}_n},L^{\mathcal{T}_n^0}))$, we may and will assume that the convergence (2.20) holds a.s. along the sequence $(n_k)_{k\geq 1}$. From the definition of $D^0(s,t)$ and the bound (2.19), we obtain that for every $(s,t)\in[0,1]^2$,

$$D(s,t)\leq D^0(s,t) \tag{2.21}$$

Similarly, a passage to the limit from the identity (2.1) gives

$$D(0,t)=Z_t-\min\{Z_s:0\leq s\leq 1\}, \tag{2.22}$$

for every $t\in[0,1]$, a.s.

The function $(s,t)\mapsto D(s,t)$ is clearly symmetric and satisfies the triangle inequality since the functions d_n do. Moreover, the fact that $d_n(i,j)=0$ if $v_i^n=v_j^n$ easily implies that $D(s,t)=0$ for s,t such that $s\sim_{\mathbf{e}} t$ a.s. (see the proof of Proposition 3.3 in [47] for a similar argument). Hence $D(s,t)$ only depends on $p_{\mathbf{e}}(s)$ and $p_{\mathbf{e}}(t)$, and D can be viewed as a pseudo-metric on the CRT $\mathcal{T}_{\mathbf{e}}$, which satisfies $D(a,b)\leq D^0(a,b)$ for every $a,b\in\mathcal{T}_{\mathbf{e}}$, by (2.21). Since D verifies the triangle inequality, the latter bound also implies

$$D(a,b)\leq D^*(a,b)$$

for every $a,b\in\mathcal{T}_{\mathbf{e}}$ a.s. To complete the proof, we need the next lemma.

Lemma 2.5.1. *We have*

$$D(a,b)=D^*(a,b)$$

for every $a,b\in\mathcal{T}_{\mathbf{e}}$ a.s.

The statement of the theorem easily follows from the lemma. Indeed, we introduce a correspondence between the metric spaces $(V(\mathcal{M}_n^\bullet)\setminus\{\partial\},2^{-1/4}n^{-1/4}d_{\mathrm{gr}}^{\mathcal{M}_n^\bullet})$ and $(\mathbf{m}_\infty,D^*)$ by setting

$$\mathcal{R}_n=\{(v^n_{\lfloor nt\rfloor},\Pi(p_{\mathbf{e}}(t))):t\in[0,1]\}.$$

From the (almost sure) convergence (2.20), and the equality $D=D^*$, we easily get that the distortion of $\mathcal{R}_n$ tends to 0 as $n\to\infty$ along the sequence $(n_k)_{k\geq1}$. It follows that the random metric space $(V(\mathcal{M}_n^\bullet)\setminus\{\partial\},2^{-1/4}n^{-1/4}d_{\mathrm{gr}}^{\mathcal{M}_n^\bullet})$ converges a.s. to $(\mathbf{m}_\infty,D^*)$ as $n\to\infty$ along the sequence $(n_k)_{k\geq1}$, in the Gromov-Hausdorff sense. Clearly, this convergence still holds if we replace $V(\mathcal{M}_n^\bullet)\setminus\{\partial\}$ by $V(\mathcal{M}_n^\bullet)$. The previous discussion shows that from every sequence of integers going to infinity, we can extract a subsequence along which the convergence stated in (2.18) holds. This suffices to complete the proof of (2.18).

It only remains to prove Lemma 2.5.1.

2.5.3 Proof of Lemma 2.5.1

Here we follow closely [48, Section 8.3]. By a continuity argument, it is enough to show that if X and Y are two independent random variables uniformly distributed over $[0,1]$, which are also independent of the sequence $(\mathcal{M}_n^\bullet)_{n\geq1}$ and of the triplet $(\mathbf{e},Z,D)$, we have

$$D(p_{\mathbf{e}}(X),p_{\mathbf{e}}(Y))=D^*(p_{\mathbf{e}}(X),p_{\mathbf{e}}(Y))\text{ a.s.}$$

Since one already knows that

$$D(p_{\mathbf{e}}(X), p_{\mathbf{e}}(Y)) \leq D^*(p_{\mathbf{e}}(X), p_{\mathbf{e}}(Y)),$$

it is enough to prove that these two random variables have the same distribution.

First, the distribution of $D^*(p_{\mathbf{e}}(X), p_{\mathbf{e}}(Y))$ can be found in [48, Corollary 7.3]:

$$D^*(p_{\mathbf{e}}(X), p_{\mathbf{e}}(Y)) \stackrel{(d)}{=} Z_X - \min\{Z_s : 0 \leq s \leq 1\}. \tag{2.23}$$

We then want to determine the distribution of $D(p_{\mathbf{e}}(X), p_{\mathbf{e}}(Y)) = D(X, Y)$. We set for $n \geq 1$,

$$i_n = \lfloor nX \rfloor \ , \ j_n = \lfloor nY \rfloor.$$

The random variables i_n and j_n are independent, independent of $\mathcal{M}_n^\bullet$ and uniformly distributed over $\{0, \ldots, n-1\}$. As we already explained in subsection 2.2, every integer between 0 and $n-1$ corresponds to a corner of a white vertex in the tree $\mathcal{T}_n$, and thus by the BDG bijection to an edge of $\mathcal{M}_n^\bullet$. We introduce a new planar map $\mathcal{M}_n^{\bullet\prime}$ in $\mathbf{M}_n^{b\bullet}$ defined by saying that $\mathcal{M}_n^{\bullet\prime}$ has the same vertices, edges, faces and origin vertex as $\mathcal{M}_n^\bullet$, but a different root edge, which is the edge associated with the corner corresponding to i_n in the BDG bijection between $\mathcal{T}_n$ and $\mathcal{M}_n^\bullet$. The orientation of this root edge is chosen with probability $\frac{1}{2}$ among the two possible ones. Since what we have done is just replacing the root edge by another oriented edge chosen uniformly at random over the $2n$ possible choices, it is easy to see that the map $\mathcal{M}_n^{\bullet\prime}$ is also uniformly distributed over $\mathbf{M}_n^{b\bullet}$.

The tree associated with $\mathcal{M}_n^{\bullet\prime}$ via the BDG bijection is denoted by $\mathcal{T}'_n$. We let $v_0'^n, \ldots, v_n'^n$ be the white contour sequence of $\mathcal{T}'_n$ and we also let d'_n be the analog of d_n when $\mathcal{M}_n^\bullet$ is replaced by $\mathcal{M}_n^{\bullet\prime}$.

Let $k_n \in \{0, \ldots, n-1\}$ be the index of the white corner of $\mathcal{T}'_n$ corresponding via the BDG bijection to the edge of $\mathcal{M}_n^\bullet$ starting from the corner j_n in $\mathcal{T}_n$. Conditionally on the pair $(\mathcal{M}_n^\bullet, \mathcal{M}_n^{\bullet\prime})$, the latter edge is uniformly distributed over the set of all edges of $\mathcal{M}_n^\bullet$ (thus also over the set of all edges of $\mathcal{M}_n^{\bullet\prime}$). It follows that, conditionally to $(\mathcal{M}_n^\bullet, \mathcal{M}_n^{\bullet\prime})$, the index k_n is uniformly distributed over $\{0, \ldots, n-1\}$, so it is independent of $\mathcal{M}_n^{\bullet\prime}$. From the definition of $\mathcal{M}_n^{\bullet\prime}$, the vertex $v_{i_n}^n$ is either equal or adjacent to $v_0'^n$ and in a similar way the vertex $v_{j_n}^n$ is either equal or adjacent to $v_{k_n}'^n$. This leads to the bound.

$$|d_n(i_n, j_n) - d'_n(0, k_n)| \leq 2. \tag{2.24}$$

Moreover we observe that

$$d'_n(0, k_n) \stackrel{(d)}{=} d_n(0, i_n) \tag{2.25}$$

because k_n is independent of $\mathcal{M}_n^{\bullet\prime}$ and uniformly distributed over $\{0, \ldots, n-1\}$, and i_n satisfies the same properties with respect to $\mathcal{M}_n^\bullet$. We now use the a.s. convergence (2.20) to get

$$2^{-1/4} n^{-1/4} d_n(0, i_n) \underset{n\to\infty}{\longrightarrow} D(0, X) = Z_X - \min\{Z_s : 0 \leq s \leq 1\}, \tag{2.26}$$

where the last equality holds by (2.22), and

$$2^{-1/4} n^{-1/4} d_n(i_n, j_n) \underset{n\to\infty}{\longrightarrow} D(X, Y). \tag{2.27}$$

Both (2.26) and (2.27) hold a.s along the subsequence $(n_k)_{k\geq 1}$. On the other hand, (2.24) and (2.25) show that the limit in (2.26) must have the same distribution as the limit in (2.27), and we get

$$D(X, Y) \stackrel{(d)}{=} Z_X - \min\{Z_s : 0 \leq s \leq 1\}.$$

Recalling (2.23), we see that $D(p_{\mathbf{e}}(X), p_{\mathbf{e}}(Y))$ and $D^*(p_{\mathbf{e}}(X), p_{\mathbf{e}}(Y))$ have the same distribution, which completes the proof of Lemma 2.5.1.

2.6 Convergence of rooted maps

In this section, we derive Theorem 2.1.1 from the convergence (2.18) for rooted and pointed maps. Notice that similar arguments appear in [11, Proposition 4]. As previously, $\mathcal{M}_n^\bullet$ is uniformly distributed over $\mathbf{M}_n^{b\bullet}$, but it will be sometimes be convenient to view $\mathcal{M}_n^\bullet$ as a random element of $\mathbf{M}_n^b$, just by "forgetting" the distinguished vertex. In particular, if F is a function on $\mathbf{M}_n^b$, the notation $F(\mathcal{M}_n^\bullet)$ means that we apply F to the rooted map obtained by forgetting the distinguished vertex of $\mathcal{M}_n^\bullet$. Similarly, we will write $\mu_n^\bullet$ for the law of $\mathcal{M}_n^\bullet$ viewed as a random element of $\mathbf{M}_n^b$. The notation μ_n will then stand for the law of $\mathcal{M}_n$, that is, the uniform probability measure on $\mathbf{M}_n^b$. Let $\|.\|$ stand for the total variation norm. In order to get Theorem 2.1.1 from (2.18), it is sufficient to prove the following result.

Proposition 2.6.1. *The following convergence holds.*

$$\|\mu_n - \mu_n^\bullet\| \underset{n\to\infty}{\longrightarrow} 0.$$

Proof. We have

$$\|\mu_n - \mu_n^\bullet\| = \frac{1}{2}\sup_{-1\le F\le 1} |E(F(\mathcal{M}_n)) - E(F(\mathcal{M}_n^\bullet))|,$$

where the supremum is over all functions $F : \mathbf{M}_n^{b\bullet} \longrightarrow [-1,1]$. The quantity $E(F(\mathcal{M}_n^\bullet))$ can be expressed in terms of $E(F(\mathcal{M}_n))$ as

$$E(F(\mathcal{M}_n^\bullet)) = \frac{E(F(\mathcal{M}_n)\mathrm{Card}\, V(\mathcal{M}_n))}{E(\mathrm{Card}\, V(\mathcal{M}_n))},$$

which implies

$$E(F(\mathcal{M}_n)) = E\left(\frac{F(\mathcal{M}_n^\bullet)}{\mathrm{Card}\, V(\mathcal{M}_n^\bullet)}\right)\frac{1}{E(1/\mathrm{Card}\, V(\mathcal{M}_n^\bullet))}. \tag{2.28}$$

We then need an estimate of $\mathrm{Card}\, V(\mathcal{M}_n^\bullet)$, which is given by the next lemma.

Lemma 2.6.2. *Let $\delta > 0$. There exists a positive constant C_δ such that*

$$P\left(\left|\mathrm{Card}\, V(\mathcal{M}_n^\bullet) - \frac{2n}{3}\right| > \delta n\right) \le \exp(-C_\delta n)$$

for all n sufficiently large.

Proof. We start by observing that the number $\mathrm{Card}\, V(\mathcal{M}_n^\bullet)$ corresponds via the BDG bijection to (1 plus) the number of white vertices of a two-type Galton-Watson tree with offspring distributions (μ_0, μ_1) given by Proposition 2.3.1, conditioned to have n edges.

Let us consider a sequence of independent two-type Galton-Watson trees with offspring distributions (μ_0, μ_1). Suppose that the white vertices of these trees are listed in lexicographical order for each tree, one tree after another, and write $A_1, A_2, \ldots$ for the respective numbers of black children of the white vertices in this enumeration. Then $A_1, A_2, \ldots$ are i.i.d random variables with distribution μ_0, and we recall that μ_0 is a geometric distribution with mean $\frac{1}{2}$. We can apply Cramer's theorem to get the exponential bound, for every $n \ge 1$,

$$P\left(\left|\frac{A_1+\cdots+A_n}{n} - \frac{1}{2}\right| \geqslant \delta\right) \leqslant \exp(-K_\delta n) \tag{2.29}$$

where K_δ is a positive constant.

Let N_0 and N_1 be respectively the numbers of white and black vertices in the first tree in our sequence, and let $N = N_0 + N_1 - 1$, which is the number of edges of this tree. The point now is the fact that if we condition on the event $\{N = n\}$, the planar map associated with the first tree becomes uniform on $\mathbf{M}_n^{b\bullet}$. Since this planar map has $N_0 + 1$ vertices, the result of the lemma will follow if we can prove that, for n sufficiently large,

$$P\Big[\Big|N_0 - \frac{2}{3}(n+1)\Big| > \delta(n+1) \,\Big|\, N = n\Big] \le \exp(-C_\delta n)$$

for some positive constant C_δ.

Recall from (2.9) that $n^{3/2}P(N = n) \longrightarrow (\sigma\sqrt{2\pi})^{-1}$ as $n \to \infty$. Therefore the preceding exponential bound will follow if we can verify that for all n large enough,

$$P\Big[\Big\{\Big|N_0 - \frac{2}{3}(n+1)\Big| > \delta(n+1)\Big|\Big\} \cap \{N = n\}\Big] \le \exp(-c_\delta n)$$

with some positive constant c_δ.

We first observe that the event $\mathcal{E}_1 := \{N_0 - \frac{2}{3}(n+1) > \delta(n+1)\} \cap \{N = n\}$ is contained in

$$\Big\{n + 1 \geqslant N_0 > \Big(\frac{2}{3} + \delta\Big)(n+1)\Big\} \cap \Big\{\frac{N_1}{N_0} < \frac{\frac{1}{3} - \delta}{\frac{2}{3} + \delta}\Big\}.$$

Therefore if we set $a_\delta = (\frac{1}{3} - \delta)/(\frac{2}{3} + \delta) < \frac{1}{2}$, the event $\mathcal{E}_1$ may only hold if, for some k such that $(\frac{2}{3} + \delta)(n+1) < k \le n+1$, the first k white vertices of our sequence of trees have less than $a_\delta k$ black children. Using (2.29), we obtain that

$$P(\mathcal{E}_1) \le \sum_{(\frac{2}{3}+\delta)(n+1)<k\le n+1} \exp(-K'_\delta k) \le \exp(-c'_\delta n)$$

for some positive constants K'_δ and c'_δ. Similar arguments give an analogous exponential bound for the probability of the event $\mathcal{E}_2 := \{N_0 - \frac{2}{3}(n+1) < -\delta(n+1)\} \cap \{N = n\}$. This completes the proof of the lemma. $\square$

Set $X_n = (2n/3)^{-1}\,\mathrm{Card}\,V(\mathcal{M}_n^\bullet)$ for every $n \ge 1$.

Lemma 2.6.3. *The random variables X_n^{-1} converge to 1 in L^1 when n tends to infinity.*

Proof. First, as $\mathrm{Card}\,V(\mathcal{M}_n^\bullet) \ge 1$, we have $X_n^{-1} \le \frac{2n}{3}$. Let $\delta > 0$. The event $\{|X_n^{-1} - 1| > \delta\}$ is contained in $\{X_n < \frac{1}{2}\} \cup \{|X_n - 1| > \frac{\delta}{2}\}$. This leads to

$$E(|X_n^{-1} - 1|) \le \delta + E(|X_n^{-1} - 1|\mathbf{1}_{\{|X_n^{-1}-1|>\delta\}}) \le \delta + \frac{2n}{3}P\Big(|X_n - 1| > \frac{\delta}{2} \wedge \frac{1}{2}\Big).$$

Hence, by Lemma 2.6.2,

$$\limsup_{n\to\infty} E(|X_n^{-1} - 1|) \le \delta$$

and the desired result follows since δ was arbitrary. $\square$

Finally we use (2.28) and Lemma 2.6.3 to get

$$\begin{aligned}
\|\mu_n - \mu_n^\bullet\| &= \frac{1}{2} \sup_{-1\leq F\leq 1} \left| E\left[F(\mathcal{M}_n^\bullet)\left(1 - \frac{1}{\operatorname{Card} V(\mathcal{M}_n^\bullet)} \frac{1}{E(1/\operatorname{Card} V(\mathcal{M}_n^\bullet))}\right)\right]\right| \\
&\leq E\left[\left|1 - \frac{1}{\operatorname{Card} V(\mathcal{M}_n^\bullet)} \frac{1}{E(1/\operatorname{Card} V(\mathcal{M}_n^\bullet))}\right|\right] \\
&= E\left[\left|1 - \frac{1/X_n}{E(1/X_n)}\right|\right] \\
&\xrightarrow[n\to\infty]{} 0.
\end{aligned}$$

This completes the proof of Proposition 2.6.1.

□

Acknowledgement. I am deeply indebted to Jean-François Le Gall for suggesting me to study this problem, for stimulating discussions and for carefully reading the manuscript and making many useful suggestions. I am grateful to Jérémie Bettinelli for a very interesting discussion concerning Proposition 2.6.1. I also thank the referee of this article for his careful reading and several helpful suggestions.

Chapitre 3

Excursion theory for Brownian motion indexed by the Brownian tree

Cette partie correspond à un travail réalisé en collaboration avec Jean-François Le Gall, soumis pour publication [2].

Contenu de ce chapitre

Abstract

We develop an excursion theory for Brownian motion indexed by the Brownian tree, which in many respects is analogous to the classical Itô theory for linear Brownian motion. Each excursion is associated with a connected component of the complement of the zero set of the tree-indexed Brownian motion. Each such connected component is itself a continuous tree, and we introduce a quantity measuring the length of its boundary. The collection of boundary lengths coincides with the collection of jumps of a continuous-state branching process with branching mechanism $\psi(u) = \sqrt{8/3}\, u^{3/2}$. Furthermore, conditionally on the boundary lengths,

the different excursions are independent, and we determine their conditional distribution in terms of an excursion measure $\mathbb{M}_0$ which is the analog of the Itô measure of Brownian excursions. We provide various descriptions of the excursion measure $\mathbb{M}_0$, and we also determine several explicit distributions, such as the joint distribution of the boundary length and the mass of an excursion under $\mathbb{M}_0$. We use the Brownian snake as a convenient tool for defining and analysing the excursions of our tree-indexed Brownian motion.

3.1 Introduction

The concept of Brownian motion indexed by a Brownian tree has appeared in various settings in the last 25 years. The Brownian tree of interest here is the so-called CRT (Brownian Continuum Random Tree) introduced by Aldous [4, 6], or more conveniently a scaled version of the CRT with a random "total mass". The CRT is a universal model for a continuous random tree, in the sense that it appears as the scaling limit of many different classes of discrete random trees (see in particular [6, 31, 66]), and of other discrete random structures (see the recent papers [23, 61]). At least informally, the meaning of Brownian motion indexed by the Brownian tree should be clear: Labels, also called spatial positions, are assigned to the vertices of the tree, in such a way that the root has label 0 and labels evolve like linear Brownian motion when moving away from the root along a geodesic segment of the tree, and of course the increments of the labels along disjoint segments are independent. Combining the branching structure of the CRT with Brownian displacements led Aldous to introduce the Integrated Super-Brownian Excursion or ISE [7], which is closely related with the canonical measures of super-Brownian motion. On the other hand, the desire to get a better understanding of the historical paths of superprocesses motivated the definition of the so-called Brownian snake [40], which is a Markov process taking values in the space of all finite paths. Roughly speaking, the value of the Brownian snake at time s is the path recording the spatial positions along the ancestral line of the vertex visited at the same time s in the contour exploration of the Brownian tree. One may view the Brownian snake as a convenient representation of Brownian motion indexed by a Brownian tree, avoiding the technical difficulty of dealing with a random process indexed by a random set.

The preceding concepts have found many applications. The Brownian snake has proved a powerful tool in the study of sample path properties of super-Brownian motion and of its connections with semilinear partial differential equations [46, 44]. ISE, and more generally Brownian motion indexed by a Brownian tree, also appear in the scaling limits of various models of statistical mechanics above the critical dimension, including lattice trees [27], percolation [33] or oriented percolation [34]. More recently, scaling limits of large random planar maps have been described by the so-called Brownian map [48, 58], which is constructed as a quotient space of the CRT for an equivalence relation defined in terms of Brownian labels assigned to the vertices of the CRT.

Our main goal in this work is to show that a very satisfactory excursion theory can be developed for Brownian motion indexed by a Brownian tree, or equivalently for the Brownian snake, which in many aspects resembles the classical excursion theory for linear Brownian motion due to Itô [35]. We also expect the associated excursion measure to be an interesting probabilistic object, which hopefully will have significant applications in related fields.

Let us give an informal description of the main results of our study. The underlying Brownian tree that we consider is denoted by $\mathcal{T}_\zeta$, for the tree coded by a Brownian excursion $(\zeta_s)_{s\geq 0}$ under the classical Itô excursion measure (see subsection 3.2.1 for more details about this coding). The tree $\mathcal{T}_\zeta$ may be viewed as a scaled version of the CRT, for which $(\zeta_s)_{s\geq 0}$ would be a Brownian excursion with duration 1. This tree is rooted at a particular vertex ρ. We write V_u for the Brownian label assigned to the vertex u of $\mathcal{T}_\zeta$. As explained above the collection $(V_u)_{u\in\mathcal{T}_\zeta}$ should be interpreted as Brownian motion indexed by $\mathcal{T}_\zeta$, starting from 0 at the root ρ. Similarly as in the case of linear Brownian motion, we may then consider the connected components of the open set

$$\{u \in \mathcal{T}_\zeta : V_u \neq 0\},$$

which we denote by $(\mathcal{C}_i)_{i\in I}$. Of course these connected components are not intervals as in the classical case, but they are connected subsets of the tree $\mathcal{T}_\zeta$, and thus subtrees of this tree. One then considers, for each component $\mathcal{C}_i$, the restriction $(V_u)_{u\in\mathcal{C}_i}$ of the labels to $\mathcal{C}_i$, and this restriction again yields a random process indexed by a continuous random tree, which we call the excursion E_i. Our main results completely determine the law of the collection $(E_i)_{i\in I}$. A first important ingredient of this description is an infinite excursion measure $\mathbb{M}_0$, which plays a similar role as the Itô excursion measure in the classical setting, in the sense that $\mathbb{M}_0$ describes the distribution of a typical excursion E_i (this is a little informal as $\mathbb{M}_0$ is an infinite measure). We can then completely describe the law of the collection $(E_i)_{i\in I}$ using the measure $\mathbb{M}_0$ and an independence property analogous to the classical setting. For this description, we first need to introduce a quantity $\mathcal{Z}_i$, called the exit measure of E_i, that measures the size of the boundary of $\mathcal{C}_i$: Note that in the classical setting the boundary of an excursion interval just consists of two points, but here of course the boundary of $\mathcal{C}_i$ is much more complicated. Furthermore, one can define, for every $z \geq 0$, a conditional probability measure $\mathbb{M}_0(\cdot \mid \mathcal{Z} = z)$ which corresponds to the law of an excursion conditioned to have boundary size z (this is somehow the analog of the Itô measure conditioned to have a fixed duration in the classical setting). Finally, we can introduce a "local time exit process" $(\mathcal{X}_t)_{t\geq 0}$ such that, for every $t > 0$, $\mathcal{X}_t$ measures the quantity of vertices u of the tree $\mathcal{T}_\zeta$ with label 0 and such that the total accumulated local time at 0 of the label process along the geodesic segment between ρ and u is equal to t. The distribution of $(\mathcal{X}_t)_{t>0}$ is known explicitly and can be interpreted as an excursion measure for the continuous-state branching process with stable branching mechanism $\psi(\lambda) = \sqrt{8/3}\,\lambda^{3/2}$. With all these ingredients at hand, we can complete our description of the distribution of the collection of excursions: Excursions E_i are in one-to-one correspondence with jumps of the local time exit process $(\mathcal{X}_t)_{t\geq 0}$, in such a way that, for every $i \in I$, the boundary length $\mathcal{Z}_i$ of E_i is equal to the size z_i of the corresponding jump, and furthermore, conditionally on the process $(\mathcal{X}_t)_{t\geq 0}$, the excursions E_i, $i \in I$ are independent, and, for every fixed j, E_j is distributed according to $\mathbb{M}_0(\cdot \mid \mathcal{Z} = z_j)$. There is a striking analogy with the classical setting (see e.g. [63, Chapter XII]), where excursions of linear Brownian excursion are in one-to-one correspondance with jumps of the inverse local time process, and the distribution of an excursion corresponding to a jump of size ℓ is the Itô measure conditioned to have duration equal to ℓ.

The preceding discussion is somewhat informal, in particular because we did not give a mathematically precise definition of the excursions E_i. It would be possible to view these excursions as random elements of the space of all "spatial trees" in the terminology of [30] (compact $\mathbb{R}$-trees $\mathcal{T}$ equipped with a continuous mapping $\phi : \mathcal{T} \to \mathbb{R}$) but for technical reasons we prefer to use the Brownian snake approach. We now describe this approach in order to give a more precise formulation of our results. Let $\mathcal{W}_0$ stand for the set of all finite real paths started from 0. Here a finite path is just a continuous function $\mathrm{w} : [0, \zeta] \to \mathbb{R}$, where $\zeta = \zeta_{(\mathrm{w})} \geq 0$ depends on w and is called the lifetime of w, and, for every $\mathrm{w} \in \mathcal{W}_0$, we write $\hat{\mathrm{w}} = \mathrm{w}(\zeta_{(\mathrm{w})})$ for the endpoint of w. The Brownian snake (with initial point 0) is the continuous Markov process $(W_s)_{s\geq 0}$ with values in $\mathcal{W}_0$ whose distribution is characterized as follows:

(i) The lifetime process $(\zeta_{(W_s)})_{s\geq 0}$ is a reflected Brownian motion on $\mathbb{R}_+$.

(ii) Conditionally on $(\zeta_{(W_s)})_{s\geq 0}$, $(W_s)_{s\geq 0}$ is time-inhomogeneous Markov, with transition kernels specified as follows: for $0 \leq s < s'$,

- $W_{s'}(t) = W_s(t)$ for every $0 \leq t \leq m(s,s') := \min\{\zeta_{(W_r)} : s \leq r \leq s'\}$;
- conditionally on W_s, $(W_{s'}(m(s,s')+t), 0 \leq t \leq \zeta_{(W_{s'})} - m(s,s'))$ is a linear Brownian motion started from $W_s(m(s,s'))$, on the time interval $[0, \zeta_{(W_{s'})} - m(s,s')]$.

We may and will assume that $(W_s)_{s\geq 0}$ is the canonical process on the space $C(\mathbb{R}_+, \mathcal{W}_0)$ of all continuous mappings from $\mathbb{R}_+$ into $\mathcal{W}_0$, and we write $\zeta_s = \zeta_{(W_s)}$ to simplify notation. Informally, the value W_s of the Brownian snake at time s is a random path with lifetime ζ_s evolving like reflected Brownian motion on $\mathbb{R}_+$. When ζ_s decreases, the path is erased from its tip, and when ζ_s increases, the path is extended by adding "little pieces" of Brownian paths at its tip.

The trivial path with lifetime 0 in $\mathcal{W}_0$ is a regular recurrent point for the process $(W_s)_{s\geq 0}$, and thus we can introduce the associated excursion measure $\mathbb{N}_0$, which is called the Brownian snake excursion measure (from 0). This is a σ-finite measure on the space $C(\mathbb{R}_+, \mathcal{W}_0)$, which can be described via properties analogous to (i) and (ii), with the only difference that in (i) the law of reflecting Brownian motion is replaced by the Itô measure of positive excursions of linear Brownian motion. In particular, the tree $\mathcal{T}_\zeta$ coded by $(\zeta_s)_{s\geq 0}$ has under $\mathbb{N}_0$ the distribution prescribed in the above informal discussion. Recall that the coding of $\mathcal{T}_\zeta$ involves a canonical projection $p_\zeta : [0,\sigma] \to \mathcal{T}_\zeta$, where $\sigma = \sup\{s \geq 0 : \zeta_s > 0\}$ (see [49, Section 3.2] or subsection 3.2.1 below). Then, the Brownian labels $(V_u)_{u\in\mathcal{T}_\zeta}$ are generated by taking $V_u = \hat{W}_s$, where $s \in [0,\sigma]$ is any instant such that $p_\zeta(s) = u$. Furthermore, the whole path W_s records the values of labels along the geodesic segment from the root ρ to u, and we sometimes say that W_s is the historical path of u.

The Brownian snake construction allows us to give a convenient representation for the excursions $(E_i)_{i\in I}$ discussed above. We observe that the connected components $(\mathcal{C}_i)_{i\in I}$ of $\{u \in \mathcal{T}_\zeta : V_u \neq 0\}$ are in one-to-one correspondence with the collection $(u_i)_{i\in I}$ of all vertices of $\mathcal{T}_\zeta$ such that

(a) $V_u = 0$;

(b) u has a strict descendant v such that labels along the geodesic segment from u to v do not vanish except at u.

The correspondence is made explicit by saying that $\mathcal{C}_i$ consists of all strict descendants v of u_i such that property (b) holds (with $u = u_i$). Then, for every $i \in I$, there are exactly two times $0 < a_i < b_i < \sigma$ such that $p_\zeta(a_i) = p_\zeta(b_i) = u_i$, and we define, for every $s \geq 0$, a random finite path $W_s^{(u_i)}$, with lifetime $\zeta_s^{(u_i)} = \zeta_{(a_i+s)\wedge b_i} - \zeta_{a_i}$, by setting

$$W_s^{(u_i)}(t) = W_{(a_i+s)\wedge b_i}(\zeta_{a_i} + t)\ , \quad 0 \leq t \leq \zeta_s^{(u_i)}.$$

The endpoints $\hat{W}_s^{(u_i)}$ of the paths $W_s^{(u_i)}$ correspond to the labels of all descendants of u_i in $\mathcal{T}_\zeta$. In fact, we are only interested in those descendants of u_i that belong to $\mathcal{C}_i$, and for this reason we introduce the following time change

$$\tilde{W}_s^{(u_i)} = W^{(u_i)}_{\eta_s^{(u_i)}}$$

where, for every $s \geq 0$,

$$\eta_s^{(u_i)} := \inf\{r \geq 0 : \int_0^r \mathrm{d}t\, \mathbf{1}_{\{\tau_0^*(W_t^{(u_i)}) \geq \zeta_t^{(u_i)}\}} > s\},$$

with the notation $\tau_0^*(\mathrm{w}) := \inf\{t > 0 : \mathrm{w}(t) = 0\}$ for $\mathrm{w} \in \mathcal{W}_0$. The effect of this time change is to eliminate the paths $W_s^{(u_i)}$ that return to 0 and survive for a positive amount of time after the return time.

Then, for every $i \in I$, the collection $(\tilde{W}_s^{(u_i)})_{s\geq 0}$, which we view as a random element of the space $C(\mathbb{R}_+, \mathcal{W}_0)$, provides a mathematically precise representation of the excursion E_i – in fact the tree $\mathcal{C}_i$ (or rather its closure in $\mathcal{T}_\zeta$) is just the tree coded by the lifetime process $(\tilde{\zeta}_s^{(u_i)})_{s\geq 0}$ of $(\tilde{W}_s^{(u_i)})_{s\geq 0}$, and the labels on $\mathcal{C}_i$ correspond in this identification to the endpoints of the paths $\tilde{W}_s^{(u_i)}$.

In order to state our first theorem, we need one more notation. For every $i \in I$, we let ℓ_i be the total local time at 0 of the historical path W_{a_i} of u_i.

Theorem 3.1.1. *There exists a σ-finite measure $\mathbb{M}_0$ on $C(\mathbb{R}_+, \mathcal{W}_0)$ such that, for any nonnegative measurable function Φ on $\mathbb{R}_+ \times C(\mathbb{R}_+, \mathcal{W}_0)$, we have*

$$\mathbb{N}_0\Bigg(\sum_{i\in I} \Phi(\ell_i, \tilde{W}^{(u_i)})\Bigg) = \int_0^\infty \mathrm{d}\ell\, \mathbb{M}_0\Big(\Phi(\ell, \cdot)\Big).$$

The reason for considering a function depending on local times should be clear from the formula of the theorem: if $\Phi(\ell, \omega)$ does not depend on ℓ, the right-hand side will be either 0 or ∞. We may write $\mathbb{M}_0$ in the form

$$\mathbb{M}_0 = \frac{1}{2}(\mathbb{N}_0^* + \check{\mathbb{N}}_0^*)$$

where $\mathbb{N}_0^*$ is supported on positive excursions and $\check{\mathbb{N}}_0^*$ is the image of $\mathbb{N}_0^*$ under $\omega \to -\omega$. Then, for every $\delta > 0$, $\mathbb{N}_0^*$ gives a finite mass to “excursions” ω that hit δ, and more precisely,

$$\mathbb{N}_0^*(\{\omega : \sup\{\hat{W}_s(\omega) : s \geq 0\} > \delta) = c_0 \delta^{-3}$$

where c_0 is an explicit constant (see Lemma 3.3.9).

In a way similar to the classical setting, one can give various representations of the measure $\mathbb{N}_0^*$. For $\varepsilon > 0$, let $\mathbb{N}_\varepsilon$ be the Brownian snake excursion measure from ε (this is just the image of $\mathbb{N}_0$ under the shift $\omega \to \varepsilon + \omega$). Consider under $\mathbb{N}_\varepsilon$ the time-changed process $\tilde{W}$ obtained by removing those paths W_s that hit 0 and then survive for a positive amount of time (this is analogous to the time change we used above to define $\tilde{W}^{(u_i)}$ from $W^{(u_i)}$). Then $\mathbb{N}_0^*$ may be obtained as the limit when $\varepsilon \to 0$ of ε^{-1} times the law of $\tilde{W}$ under $\mathbb{N}_\varepsilon$. See Theorem 3.3.7 and Corollary 3.3.10 for precise statements. This result is analogous to the classical result saying that the Itô measure of positive excursions is the limit (in a suitable sense) of $(2\varepsilon)^{-1}$ times the law of linear Brownian motion started from ε and stopped upon hitting 0.

Similarly, one can give a description of $\mathbb{N}_0^*$ analogous to the well-known Bismut decomposition for the Itô measure [63, Theorem XII.4.7]. Under $\mathbb{N}_0^*$, pick a vertex of the tree coded by $(\zeta_s)_{s\geq 0}$ according to the volume measure on this tree, re-root the tree at that vertex and shift all labels so that the label of the new root is again 0. This construction yields a new measure on $C(\mathbb{R}_+, \mathcal{W}_0)$, which turns out to be the same (up to a simple density) as the measure obtained by picking $x \leq 0$

according to Lebesgue measure on $(-\infty, 0)$ and then, under the measure $\mathbb{N}_0$ restricted to the event where one of the paths W_s hits $-x$, removing all paths W_s that go below level x. See Theorem 3.5.1 below for a more precise statement.

We now introduce exit measures under $\mathbb{M}_0$.

Proposition 3.1.2. *One can choose a sequence $(\alpha_n)_{n\geq 1}$ of positive reals converging to 0 so that, $\mathbb{M}_0$ a.e., the limit*

$$Z_0^* := \lim_{n\to\infty} \alpha_n^{-2} \int_0^\infty \mathbf{1}_{\{0<|\hat{W}_s|<\alpha_n\}}\,\mathrm{d}s$$

exists and defines a positive random variable. Furthermore, this limit does not depend on the choice of the sequence $(\alpha_n)_{n\geq 1}$.

Informally, for every $i \in I$, $Z_0^*(\tilde{W}^{(u_i)})$ counts the number of paths $\tilde{W}^{(u_i)}$ that return to 0, and thus measures the size of the boundary of $\mathcal{C}_i$. On the other hand, the quantity $\sigma(\tilde{W}^{(u_i)})$ corresponds to the volume of $\mathcal{C}_i$. Quite remarkably, one can obtain an explicit formula for the joint distribution of the pair (Z_0^*, σ) under $\mathbb{M}_0$. This distribution has density

$$f(z,s) = \frac{\sqrt{3}}{2\pi}\sqrt{z}\, s^{-5/2} \exp\left(-\frac{z^2}{2s}\right)$$

with respect to Lebesgue measure on $\mathbb{R}_+ \times \mathbb{R}_+$ (Proposition 3.6.2).

Using scaling arguments, one can then canonically define, for every $z > 0$, the conditional probability measure $\mathbb{M}_0(\cdot \mid Z_0^* = z)$, which will play an important role in our description of the distribution of the collection $(W^{(u_i)})_{i\in I}$. Before stating our theorem identifying this distribution, we need a last ingredient. For every $s \geq 0$ and $t \in [0, \zeta_s]$, write $L_t^0(W_s)$ for the local time at level 0 and at time t of the path W_s (this makes sense under the measure $\mathbb{N}_0$). We observe that, under the measure $\mathbb{N}_0$, the process

$$\mathbf{W}_s := (W_s, L^0(W_s)) = (W_s(t), L_t^0(W_s))_{0\leq t\leq \zeta_s}$$

can be viewed as the Brownian snake (under its excursion measure from $(0,0)$) associated with a spatial motion which is now the pair consisting of a linear Brownian motion and its local time at 0 (the Brownian snake associated with a Markov process is defined by properties analogous to (i) and (ii) above, with the only difference that in (ii) linear Brownian motion is replaced by the Markov process in consideration). See [44], and notice that the spatial motion used to define the Brownian snake needs to satisfy certain continuity properties which hold in the present situation. Following [44, Chapter V], we can then define, for every $r > 0$, the exit measure of $\mathbf{W}$ from the open set $O_r = \mathbb{R} \times [0, r)$, and we denote this exit measure by $\mathcal{X}_r$ – to be precise the exit measure is a measure on ∂O_r, but here it is easily seen to be concentrated on the singleton $\{0\} \times \{r\}$, and $\mathcal{X}_r$ denotes its total mass. Informally, $\mathcal{X}_r$ measures the quantity of paths W_s whose endpoint is 0 and which have accumulated a total local time at 0 equal to r.

One can explicitly determine the "law" of the exit measure process $(\mathcal{X}_r)_{r>0}$ under $\mathbb{N}_0$, using on one hand Lévy's famous theorem relating the law of the local time process of a linear Brownian motion B to that of the supremum process of B, and on the other hand known results about exit measures from intervals. This process is Markovian, with the transition mechanism of the continuous-state branching process with stable branching mechanism $\psi(\lambda) = \sqrt{8/3}\,\lambda^{3/2}$. In particular the process $(\mathcal{X}_r)_{r>0}$ has a càdlàg modification, which we consider from now on.

Recall that, for every $i \in I$, ℓ_i denotes the local time at 0 of the historical path of u_i.

Proposition 3.1.3. *The numbers ℓ_i, $i \in I$ are exactly the jump times of the process $(\mathcal{X}_r)_{r>0}$. Furthermore, for every $i \in I$, the size $Z_0^*(\tilde{W}^{(u_i)})$ of the boundary of $\mathcal{C}_i$ is equal to the jump $\Delta\mathcal{X}_{\ell_i}$.*

We can now state the main result of this introduction.

Theorem 3.1.4. *Under $\mathbb{N}_0$, conditionally on the local time exit process $(\mathcal{X}_r)_{r>0}$, the excursions $(\tilde{W}^{(u_i)})_{i\in I}$ are independent and, for every $j \in I$, the conditional distribution of $\tilde{W}^{(u_j)}$ is $\mathbb{M}_0(\cdot \mid Z_0^* = \Delta\mathcal{X}_{\ell_j})$.*

In the classical theory, the collection of excursions of linear Brownian motion is described in terms of a Poisson point process. Such a representation is also possible here and the relevant Poisson point process is linked with the Poisson process of jumps of the Lévy process that corresponds to the continuous-state branching process $\mathcal{X}$ via Lamperti's transformation. We refrained from explaining this representation in this introduction because the formulation is somewhat more intricate than in the classical case (see however Proposition 3.7.5) and requires to add extra randomness to get a complete construction of the Poisson point process.

Let us make a few remarks. First, although we stated our main results under the infinite measure $\mathbb{N}_0$, one can give equivalent statements in the more familiar setting of probability measures, for instance by conditioning $\mathbb{N}_0$ on specific events with finite mass (such as the event where at least one of the paths W_s has accumulated a total local time at 0 greater than δ, for some fixed $\delta > 0$) or by dealing with a Poisson measure with intensity $\mathbb{N}_0$ – such Poisson measures are in fact needed when one studies the connections between the Brownian snake and superprocesses. The second remark is that we could have considered excursions away from $a \neq 0$ instead of the particular case $a = 0$. There is a minor difference, due to the special connected component of $\{u \in \mathcal{T}_\zeta : V_u \neq a\}$ that contains the root. The study of the connected components other than the special one can be reduced to the case $a = 0$ by an application of the so-called special Markov property (see subsection 3.2.4). As a last and important remark, most of the following proofs and statements deal with excursions "above the minimum" (see Section 3.3 for the definition) and not with the excursions away from 0 that we considered in this introduction. However the results about excursions away from 0 can then be derived using the already mentioned theorem of Lévy, and we explain this derivation in detail in Section 3.8. The reason for considering first excursions above the minimum comes from the fact that certain technical details become significantly simpler. In particular, the local time exit process is replaced by the more familiar process of exit measures from intervals.

An important motivation for the present work comes from the construction of the Brownian map as a quotient space of the CRT for an equivalence relation defined in terms of Brownian motion indexed by the CRT (see e.g. [48, Section 2.5]). The recent paper [25] discusses the infinite volume version of the Brownian map called the Brownian plane. In a way similar to the Brownian map, the Brownian plane is obtained as a quotient space of a tree consisting of an infinite spine and a Poisson collection of Brownian trees branching off the spine, and this tree is equipped with Brownian labels corresponding to distances from the root in the Brownian plane. The main goal of [25] is to study the process of hulls, where, for every $r > 0$, the hull of radius r is obtained by filling in the bounded holes in the ball of radius r centered at the root vertex of the Brownian plane. It turns out that discontinuities of the process of hulls correspond to excursions above the minimum for one of the Brownian motions indexed by the Brownian trees branching off the spine (this can be seen from formula (16) of [25]). Such a discontinuity appears when the hull of radius r "swallows" a connected component of the complement of the ball of radius r, and this connected component consists of the equivalent classes of the vertices belonging to the associated excursion above the minimum at level r. This relation explains why several formulas and calculations below are reminiscent of those in [25]. In particular the conditional distribution of the mass σ of an excursion given the boundary length Z_0^* (see Proposition 3.6.2) appears in [25, Theorem 1.3], as well as in the companion paper [24], where this distribution is interpreted as the limiting law of the number of faces of a Boltzmann triangulation with a boundary of fixed size.

In the same direction, there are close relations between the present article and the recent work of Miller and Sheffield [59, 60] aiming at proving the equivalence of the Brownian map and Liouville quantum gravity with parameter $\gamma = \sqrt{8/3}$. In particular, the paper [59] uses what we call Brownian snake excursions above the minimum to define the notion of a Brownian disk, corresponding to bubbles appearing in the exploration of the Brownian map: See the definition of μ^L_{DISK} in Proposition 4.4 and its proof in [59]. A key idea of [59] is the fact that one can use such Brownian disks to reconstruct the Brownian map by filling in the holes of the so-called "Lévy net", which itself corresponds to the union of the boundaries of hulls centered at the root (to be precise, the definition of hulls here requires that there is a marked vertex in addition to the root of the Brownian map).

The present paper is organized as follows. Section 2 below presents a number of preliminary observations. In contrast with the previous lines where we consider the canonical space $C(\mathbb{R}_+, \mathcal{W}_0)$, we have chosen to define the measure $\mathbb{N}_0$ on a smaller canonical space, the space of "snake trajectories" (see subsection 3.2.2). The reason for this choice is that several transformations, such as the re-rooting operation, or the truncation operation allowing us to eliminate paths W_s hitting a certain level, are more conveniently defined and analysed on this smaller space. Snake trajectories are in one-to-one correspondence with tree-like paths (also defined in subsection 3.2.2) via a homeomorphism theorem of Marckert and Mokkadem [54], and this bijection is useful to simplify certain convergence arguments. Subsection 3.2.4 gives a precise statement of the special Markov property which later plays an important role.

Section 3.3 provides a construction of the measure $\mathbb{N}_0^*$, by proving the analog of Theorem 3.1.1 for excursions above the minimum. As a by-product, this proof also yields the above-mentioned approximation of $\mathbb{N}_0^*$ in terms of the Brownian snake under $\mathbb{N}_\varepsilon$, truncated at level 0. Section 3.4 describes an almost sure version of this approximation, which is useful in further developments. Section 3.5 gives our analog of the Bismut decomposition theorem for the measure $\mathbb{N}_0^*$. The proof is based on a re-rooting invariance property of the Brownian snake which can be found in [51].

Section 3.6 contains the definition the exit measure Z_0^* under $\mathbb{N}_0^*$, and the derivation of the joint distribution of the pair (Z_0^*, σ). As an important technical ingredient of the proof of our main results, we also verify that the approximation of the measure $\mathbb{N}_0^*$ by a truncated Brownian snake under $\mathbb{N}_\varepsilon$ can be stated jointly with the convergence of the corresponding exit measures (Proposition 3.6.3). Section 3.7 contains the proof of the results analogous to Proposition 3.1.3 and Theorem 3.1.4 in the slightly different setting of excursions above the minimum. In a way very similar to the classical theory, we introduce an auxiliary Poisson point process with intensity $\mathrm{d}t \otimes \mathbb{N}_0^*(\mathrm{d}\omega)$, such that all excursions above the minimum can be recovered from the atoms of this process – but as mentioned earlier the construction of this Poisson point process is somewhat more delicate than in the classical case. Finally, Section 3.8 explains how the results of the present introduction can be derived from those concerning excursions above the minimum.

3.2 Preliminaries

3.2.1 Coding a real tree by a function

In this subsection, we recall without proof a number of simple properties of the coding of compact $\mathbb{R}$-trees by functions. We refer to [30] and [49] for additional details.

Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a nonnegative continuous function on $\mathbb{R}_+$ such that $h(0) = 0$. We assume that h has compact support, so that

$$\sigma := \sup\{t \geq 0 : h(t) > 0\} < \infty.$$

Here and later we make the convention that $\sup \varnothing = 0$.

For every $s, t \in \mathbb{R}_+$, we set

$$d_h(s,t) := h(s) + h(t) - 2 \min_{s\wedge t \leq r \leq s \vee t} h(r).$$

Then d_h is a pseudo-distance on $\mathbb{R}_+$. We introduce the associated equivalence relation on $\mathbb{R}_+$, defined by setting $s \sim_h t$ if and only if $d_h(s,t) = 0$, or equivalently

$$h(s) = h(t) = \min_{s\wedge t \leq r \leq s \vee t} h(r).$$

Then, d_h induces a distance on the quotient space $\mathbb{R}_+/ \sim_h$.

Lemma 3.2.1. *The quotient space $\mathcal{T}_h := \mathbb{R}_+/ \sim_h$ equipped with the distance d_h is a compact $\mathbb{R}$-tree called the tree coded by h. The canonical projection from $\mathbb{R}_+$ onto $\mathcal{T}_h$ is denoted by p_h.*

See e.g. [30, Theorem 2.1] for a proof of this lemma as well as for the definition of $\mathbb{R}$-trees. For every $u, v \in \mathcal{T}_h$, the segment $[\![u, v]\!]$ is defined as the range of the (unique) geodesic from u to v in $(\mathcal{T}_h, d_h)$. The sets $]\!]u, v[\![$ or $]\!]u, v]\!]$ are then defined with the obvious meaning.

Write ρ for the equivalent class of 0 in the quotient $\mathbb{R}_+/ \sim_h$, and note that, for every $s \geq 0$, $d_h(\rho, p_h(s)) = h(s)$. We call ρ the root of $\mathcal{T}_h$, and the ancestral line of a point $u \in \mathcal{T}_h$ is the geodesic segment $[\![\rho, u]\!]$. We can then define a genealogical relation on $\mathcal{T}_h$ by saying that u is an ancestor of v (or v is a descendant of u) if u belongs to $[\![\rho, v]\!]$. We will use the notation $u \prec v$ to mean that u is an ancestor of v. If $s, t \geq 0$, the property $p_h(s) \prec p_h(t)$ holds if and only if

$$h(s) = \min_{s\wedge t \leq r \leq s \vee t} h(r).$$

If $u, v \in \mathcal{T}_h$, the last common ancestor of u and v is the unique point, denoted by $u \wedge v$, such that

$$[\![\rho, u]\!] \cap [\![\rho, v]\!] = [\![\rho, u \wedge v]\!].$$

If $u = p_h(s)$ and $v = p_h(t)$ then $u \wedge v = p_h(r)$, where r is any time in $[s \wedge t, s \vee t]$ such that $h(r) = \min\{h(r') : r' \in [s \wedge t, s \vee t]\}$.

We call leaf of $\mathcal{T}_h$ any point $u \in \mathcal{T}_h$ which has no descendant other than itself. We let $\mathrm{Sk}(\mathcal{T}_h)$, the skeleton of $\mathcal{T}_h$, be the set of all points of $\mathcal{T}_h$ that are no leaves. The multiplicity of a point $u \in \mathcal{T}_h$ is the number of connected components of $\mathcal{T}_h \backslash \{u\}$. A point $u \neq \rho$ is a leaf if and only if its multiplicity is 1.

Suppose in addition that h satisfies the following properties:

(i) h does not vanish on $(0, \sigma)$;

(ii) h is not constant on any nontrivial subinterval of $(0, \sigma)$;

(iii) the local minima of h on $(0, \sigma)$ are distinct.

All these properties hold in the applications developed below, where h is a Brownian excursion away from 0. Then the multiplicity of any point of $\mathcal{T}_h$ is at most 3. Furthermore, a point u has multiplicity 3 if and only if u is the form $u = p_h(r)$ where r is a time of local minimum of h on $(0, \sigma)$. In that case there are exactly three values of s such that $p_h(s) = u$, namely $s = \sup\{t < r : h(t) > h(r\}$, $s = r$ and $s = \inf\{t > r : h(t) \leq h(r)\}$. Points of multiplicity 3 will be called branching points of $\mathcal{T}_h$. If u and v are two points of $\mathcal{T}_h$, and if $u \wedge v \neq u$ and $u \wedge v \neq v$, then $u \wedge v$ is a branching point. Finally, if u is a point of $\mathrm{Sk}(\mathcal{T}_h)$ which is not a branching point, then there are exactly two times $0 \leq s_1 < s_2 \leq \sigma$ such that $p_h(s_1) = p_h(s_2) = u$, and the descendants of u are the points $p_h(s)$ when s varies over $[s_1, s_2]$.

3.2.2 Canonical spaces for the Brownian snake

Before we recall the basic facts that we need about the Brownian snake, we start by discussing the canonical space on which this random process will be defined (for technical reasons, we choose a canonical space suitable for the definition of the Brownian snake excursion measures, which would not be appropriate for the Brownian snake starting from an arbitrary initial value as considered above in the Introduction).

Recall the notion of a finite path from the Introduction. We let $\mathcal{W}$ denote the space of all finite paths in $\mathbb{R}$, and write $\zeta_{(\mathrm{w})}$ for the lifetime of a finite path $\mathrm{w} \in \mathcal{W}$. The set $\mathcal{W}$ is a Polish space when equipped with the distance

$$d_{\mathcal{W}}(\mathrm{w}, \mathrm{w}') = |\zeta_{(\mathrm{w})} - \zeta_{(\mathrm{w}')}| + \sup_{t \geq 0} |\mathrm{w}(t \wedge \zeta_{(\mathrm{w})}) - \mathrm{w}'(t \wedge \zeta_{(\mathrm{w}')})|.$$

The endpoint or tip of the path w is denoted by $\hat{\mathrm{w}} = \mathrm{w}(\zeta_{(\mathrm{w})})$. For every $x \in \mathbb{R}$, we set $\mathcal{W}_x = \{\mathrm{w} \in \mathcal{W} : \mathrm{w}(0) = x\}$. The trivial element of $\mathcal{W}_x$ with zero lifetime is identified with the point x – in this way we view $\mathbb{R}$ as the subset of $\mathcal{W}$ consisting of all finite paths with zero lifetime. We will also use the notation $\underline{\mathrm{w}} = \min\{\mathrm{w}(t) : 0 \leq t \leq \zeta_{(\mathrm{w})}\}$.

We next turn to snake trajectories.

Definition 3.2.2. *Let $x \in \mathbb{R}$. A snake trajectory with initial point x is a continuous mapping*

$$\begin{aligned} \omega : \mathbb{R}_+ &\to \mathcal{W}_x \\ s &\mapsto \omega_s \end{aligned}$$

which satisfies the following two properties:

(i) *We have $\omega_0 = x$ and the number $\sigma(\omega) := \sup\{s \geq 0 : \omega_s \neq x\}$, called the duration of the snake trajectory ω, is finite.*

(ii) *For every $0 \leq s \leq s'$, we have*

$$\omega_s(t) = \omega_{s'}(t) \ , \quad \textit{for every } 0 \leq t \leq \min_{s \leq r \leq s'} \zeta_{(\omega_r)}.$$

We write $\mathcal{S}_x$ for the set of all snake trajectories with initial point x, and

$$\mathcal{S} := \bigcup_{x\in\mathbb{R}} \mathcal{S}_x$$

for the set of all snake trajectories. If $\omega \in \mathcal{S}$, we will often write $W_s(\omega) = \omega_s$ and $\zeta_s(\omega) = \zeta_{(\omega_s)}$ for every $s \geq 0$.

Remark. Property (ii) is called the snake property. It is not hard to verify that, for any mapping $\omega : \mathbb{R}_+ \to \mathcal{W}_x$ such that both the lifetime function $s \mapsto \zeta_s(\omega)$ and the tip function $s \mapsto \hat{W}_s(\omega)$ are continuous, then the snake property (ii) implies that ω is continuous.

The set $\mathcal{S}$ is equipped with the distance

$$d_{\mathcal{S}}(\omega, \omega') = |\sigma(\omega) - \sigma(\omega')| + \sup_{s\geq 0} d_{\mathcal{W}}(W_s(\omega), W_s(\omega')).$$

Then $(\mathcal{S}, d_{\mathcal{S}})$ is a Polish space and $\mathcal{S}_x$ is a closed subset of $\mathcal{S}$.

We will use the notation

$$\begin{aligned} \|\omega\| &= \sup\{|\omega_s(t)| : s \geq 0, 0 \leq t \leq \zeta_s(\omega)\} = \sup\{|\hat{\omega}_s| : s \geq 0\}, \\ M(\omega) &= \sup\{\omega_s(t) : s \geq 0, 0 \leq t \leq \zeta_s(\omega)\} = \sup\{\hat{\omega}_s : s \geq 0\}, \end{aligned}$$

for $\omega \in \mathcal{S}$. The fact that the two suprema in the definition of $\|\omega\|$ (or in the definition of $M(\omega)$) are equal is a simple consequence of the snake property, which implies that

$$\{\omega_s(t) : s \geq 0, 0 \leq t \leq \zeta_s(\omega)\} = \{\hat{\omega}_s : s \geq 0\}.$$

One easily checks that a snake trajectory ω is completely determined by the two functions $s \mapsto \zeta_s(\omega)$ and $s \to \hat{W}_s(\omega)$. We will state this in a more precise form, but for this we first need to introduce tree-like paths.

Definition 3.2.3. *A tree-like path is a pair (ζ, f) where $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ and $f : \mathbb{R}_+ \to \mathbb{R}$ are continuous functions that satisfy the following properties:*

(i) *We have $\zeta(0) = 0$ and the number $\sigma(\zeta) := \sup\{s \geq 0 : \zeta(s) \neq 0\}$ is finite.*

(ii) *For every $0 \leq s \leq s'$, the condition*

$$\zeta(s) = \zeta(s') = \min_{s\leq r\leq s'} \zeta(r)$$

implies that $f(s) = f(s')$.

The set of all tree-like paths is denoted by $\mathbb{T}$, and, for every $x \in \mathbb{R}$, $\mathbb{T}_x := \{(\zeta, f) \in \mathbb{T} : f(0) = x\}$ denotes the set of all tree-like paths with initial point x.

Remark. Our terminology is inspired by the work of Hambly and Lyons who give a slightly different definition of a tree-like path in a more general setting (see [32, Definition 1.2]).

It follows from property (ii) that, if $(\zeta, f) \in \mathbb{T}_x$, we have $f(s) = x$ for every $s \geq \sigma(\zeta)$. The set $\mathbb{T}$ is equipped with the distance

$$d_{\mathbb{T}}((\zeta, f), (\zeta', f')) = |\sigma(\zeta) - \sigma(\zeta')| + \sup_{s\geq 0}(|\zeta(s) - \zeta'(s)| + |f(s) - f'(s)|).$$

If (ζ, f) is a tree-like path, ζ satisfies the assumptions required in subsection 3.2.1 to define the tree $\mathcal{T}_\zeta$. Then, property (ii) just says that, for every $s \geq 0$, $f(s)$ only depends on $p_\zeta(s)$, and thus f can as well be viewed as a function on the tree $\mathcal{T}_\zeta$. Furthermore the function induced by f on $\mathcal{T}_\zeta$ is also continuous. For $u \in \mathcal{T}_\zeta$, we then interpret $f(u)$ as a spatial position, or a label, assigned to the point u.

Proposition 3.2.4. *The mapping $\Delta : \mathcal{S} \to \mathbb{T}$ defined by $\Delta(\omega) = (\zeta, f)$, where $\zeta(s) = \zeta_s(\omega)$ and $f(s) = \hat{W}_s(\omega)$, is a homeomorphism from $\mathcal{S}$ onto $\mathbb{T}$.*

This is essentially the homeomorphism theorem of Marckert and Mokkadem [54, Theorem 2.1]. Marckert and Mokkadem impose the extra condition $\sigma = 1$ for snake trajectories and tree-like paths, but the proof is the same without this condition.

Let us briefly explain why Proposition 3.2.4 is relevant to our purposes. Much of what follows is devoted to studying the convergence of certain (random) snake trajectories. By Proposition 3.2.4, this convergence is equivalent to that of the associated tree-like paths, which is often easier to establish.

Remark. Let (ζ, f) be a tree-like path, and let ω be the associated snake trajectory. We already noticed that f can be viewed as a continuous function on the tree $\mathcal{T}_\zeta$ coded by ζ. The same holds for the mapping $s \mapsto \omega_s$. More precisely, for every $s \geq 0$, and every $t \leq \zeta_s(\omega) = \zeta(s)$, $\omega_s(t)$ is the value of f at the unique ancestor of $p_\zeta(s)$ at distance t from the root (recall that $d_\zeta(\rho, p_\zeta(s)) = \zeta(s)$). Thus the finite path $\omega_s = (\omega_s(t))_{0 \leq t \leq \zeta_s(\omega)}$ provides the values of f along the ancestral line of $p_\zeta(s)$. We say that ω_s is the historical path of $p_\zeta(s)$.

Lemma 3.2.5. *Let ω be a snake trajectory and $(\zeta, f) = \Delta(\omega)$. Let $0 < s < s' < \sigma(\omega)$ such that*

$$\zeta(s) = \zeta(s') = \min_{s \leq r \leq s'} \zeta(r).$$

Set, for every $r \geq 0$,

$$\begin{aligned} \zeta'(r) &= \zeta((s+r) \wedge s') - \zeta(s) \\ f'(r) &= f((s+r) \wedge s'). \end{aligned}$$

Then, (ζ', f') is a tree-like path and the corresponding snake trajectory $\omega' = \Delta^{-1}(\zeta', f')$ is called the subtrajectory of ω associated with the interval $[s, s']$.

We omit the easy proof. The assumption of the lemma is equivalent to saying that $p_\zeta(s) = p_\zeta(s')$. Suppose in addition that $\{r \geq 0 : p_\zeta(r) = p_\zeta(s)\} = \{s, s'\}$. Then $u := p_\zeta(s)$ is a point of multiplicity 2 of $\mathrm{Sk}(\mathcal{T}_\zeta)$, and the subtree of descendants of u is coded by f'. Furthermore the snake trajectory ω' describes the spatial positions of the descendants of u.

Let us finally introduce three useful operations on snake trajectories. The first one is just the obvious translation. If $a \in \mathbb{R}$ and $\omega \in \mathcal{S}$, $\kappa_a(\omega)$ is obtained by adding a to all paths ω_s: In other words $\zeta_s(\kappa_a(\omega)) = \zeta_s(\omega)$ and $\hat{W}_s(\kappa_a(\omega)) = \hat{W}_s(\omega) + a$ for every $s \geq 0$.

The second operation is the re-rooting operation. Let ω be a snake trajectory and let (ζ, f) be the associated tree-like path. Fix $s \in [0, \sigma]$. We will define a new snake trajectory $R_s(\omega)$, which is more conveniently described in terms of its associated tree-like path $(\zeta^{[s]}, f^{[s]}) = \Delta(R_s(\omega))$. Roughly speaking, $\zeta^{[s]}$ is the coding function for the tree $\mathcal{T}_\zeta$ re-rooted at $p_\zeta(s)$ (this is informal since the coding function of a tree is not unique) and $f^{[s]}$ describes the "same function" as f but viewed on the re-rooted tree. To make this more precise, we set for every $r \in [0, \sigma(\omega)]$,

$$\zeta^{[s]}(r) = \zeta(s \oplus r) + \zeta(s) - 2 \min_{s \wedge (s \oplus r) \leq t \leq s \vee (s \oplus r)} \zeta(t),$$

where $s \oplus r = s + r$ if $s + r \leq \sigma(\omega)$ and $s \oplus r = s + r - \sigma(\omega)$ otherwise. We also set $\zeta^{[s]}(r) = 0$ if $r > \sigma(\omega)$. Furthemore we set $f^{[s]}(r) = f(s \oplus r)$ if $r \in [0, \sigma(\omega)]$ and $f^{[s]}(r) = f(s)$ if $r > \sigma(\omega)$. See [30, Lemma 2.2] for the fact that the mapping $[0, \sigma(\omega)] \ni r \mapsto s \oplus r$ induces an isometry from the tree $\mathcal{T}_{\zeta^{[s]}}$ onto the tree $\mathcal{T}_\zeta$ (this in particular implies that $(\zeta^{[s]}, f^{[s]})$ is a tree-like path), and [51, Section 2.3] for more details about this re-rooting operation.

The third and last operation is the truncation of snake trajectories, which will be important in this work. Roughly speaking, if $\omega \in \mathcal{S}_x$ and $y \neq x$, the truncation of ω at y is the new snake trajectory ω' such that the values ω'_s are exactly all values ω_s for s such that ω_s does not hit y, or hits y for the first time at its lifetime. Let us give a more precise definition. First, for any $\mathrm{w} \in \mathcal{W}$ and $y \in \mathbb{R}$, we set

$$\tau_y(\mathrm{w}) := \inf\{t \in [0, \zeta_{(\mathrm{w})}] : \mathrm{w}(t) = y\} \ , \quad \tau_y^*(\mathrm{w}) := \inf\{t \in (0, \zeta_{(\mathrm{w})}] : \mathrm{w}(t) = y\} \ ,$$

with the usual convention $\inf \varnothing = \infty$. Note that $\tau_y^*(\mathrm{w})$ may be different from $\tau_y(\mathrm{w})$ only if $\mathrm{w}(0) = y$, but this case will be important in what follows.

Proposition 3.2.6. *Let $x, y \in \mathbb{R}$. Let $\omega \in \mathcal{S}_x$, and for every $s \geq 0$, set*

$$A_s(\omega) = \int_0^s \mathrm{d}r \, \mathbf{1}_{\{\zeta_r(\omega) \leq \tau_y^*(\omega_r)\}},$$

and

$$\eta_s(\omega) = \inf\{r \geq 0 : A_r(\omega) > s\}.$$

Then setting $\omega'_s = \omega_{\eta_s(\omega)}$ for every $s \geq 0$ defines an element of $\mathcal{S}_x$, which will be denoted by $\omega' = \mathrm{tr}_y(\omega)$ and called the truncation of ω at y.

Proof. First note that, by property (i) of the definition of a snake trajectory, we have $A_s(\omega) \to \infty$ as $s \to \infty$ (because $\zeta_r(\omega) \leq \tau_y^*(\omega_r)$ if $r \geq \sigma(\omega)$), and therefore $\eta_s(\omega) < \infty$ for every $s \geq 0$, so that the definition of ω' makes sense.

We need to verify that $\omega' \in \mathcal{S}_x$. To this end, we observe that the mapping $s \to \eta_s(\omega)$ is right-continuous with left limits given by

$$\eta_{s-}(\omega) = \inf\{r \geq 0 : A_r(\omega) = s\} \ , \quad \forall s > 0.$$

To simplify notation, we write $\eta_s = \eta_s(\omega)$, $\eta_{s-} = \eta_{s-}(\omega)$, $A_s = A_s(\omega)$ and $\zeta_s = \zeta_s(\omega)$ in what follows.

We first verify the continuity of the mapping $s \mapsto \omega'_s$. Let $s \geq 0$ such that $\zeta_{\eta_s} > 0$. By the definition of η_s there are values of $r > \eta_s$ arbitrarily close to η_s such that $\zeta_r \leq \tau_y^*(\omega_r)$. Using the snake property, it then follows that the path $(\omega_{\eta_s}(t))_{0 < t \leq \zeta_{\eta_s}}$ does not hit y, or hits y only at time ζ_{η_s} (notice that we excluded the value $t = 0$ because of the particular case $y = x$, since we have trivially $\omega_{\eta_s}(0) = y$ in that case). Similarly, for every $s > 0$ such that $\zeta_{\eta_{s-}} > 0$, the path $(\omega_{\eta_{s-}}(t))_{0 < t \leq \zeta_{\eta_{s-}}}$ does not hit y, or hits y only at time $\zeta_{\eta_{s-}}$.

Let $s > 0$ be such that $\eta_{s-} < \eta_s$. The key observation is to note that

$$\zeta_r \geq \zeta_{\eta_{s-}} = \zeta_{\eta_s} , \quad \forall r \in [\eta_{s-}, \eta_s]. \tag{3.1}$$

In fact, suppose that (3.1) fails, so that certain values of ζ on the time interval (η_{s-}, η_s) are strictly smaller that $\zeta_{\eta_s} \vee \zeta_{\eta_{s-}}$. Suppose for definiteness that $\zeta_{\eta_{s-}} \leq \zeta_{\eta_s}$ (the other case $\zeta_{\eta_{s-}} \geq \zeta_{\eta_s}$ is treated similarly). Then we can find $r \in (\eta_{s-}, \eta_s)$ such that $0 < \zeta_r < \zeta_{\eta_s}$ and $\zeta_r = \min\{\zeta_u : u \in [r, \eta_s]\}$. By the snake property this means that ω_r is the restriction of ω_{η_s} to $[0, \zeta_r]$, and, since we know that $(\omega_{\eta_s}(t))_{0<t<\zeta_{\eta_s}}$ does not hit y, it follows that $\tau_y^*(\omega_r) = \infty$. Hence we have also $\tau_y^*(\omega_{r'}) = \infty$, for all r' sufficiently close to r, and therefore $A_{\eta_s} > A_{\eta_{s-}}$, which is a contradiction.

The mapping $s \mapsto \omega_{\eta_s}$ is right-continuous and its left limit at $s > 0$ is $\omega_{\eta_{s-}}$. Property (3.1) and the snake property show that, for every s such that $\eta_{s-} < \eta_s$, we have $\omega_{\eta_{s-}} = \omega_{\eta_s}$, so that the mapping $s \mapsto \omega_{\eta_s} = \omega'_s$ is continuous.

Furthermore, it also follows from (3.1) that, for every $s \leq s'$,

$$\min_{r \in [s,s']} \zeta_{\eta_r} = \min_{r \in [\eta_s, \eta_{s'}]} \zeta_r$$

and the snake property for ω' is a consequence of the one for ω.

Finally, we also need to verify that $\omega'_0 = x$. This is immediate if $y \neq x$ (because clearly $\eta_0 = 0$ in that case) but an argument is required in the case $y = x$, which we consider now. It suffices to verify that $\zeta_{\eta_0} = 0$. We argue by contradiction and assume that $\zeta_{\eta_0} > 0$, which implies that $\eta_0 > 0$. By previous observations, the path ω_{η_0} does not hit x during the time interval $(0, \zeta_{\eta_0})$. However, by the snake property again, this implies that there is a set of positive Lebesgue measure of values of $r \in (0, \eta_0)$ such that $\tau_x^*(\omega_r) = \infty$, which contradicts the definition of η_0. $\square$

Remark. If $s > 0$ is such that $\eta_{s-} < \eta_s$, and furthermore $\zeta_{\eta_s} > 0$, then we have $\tau_y^*(\omega_{\eta_s}) = \zeta_s$. Indeed, since $A_{\eta_s} = A_{\eta_{s-}} = s$, there exist values of $r < \eta_s$ arbitrarily close to η_s such that $\tau_y^*(\omega_r) < \zeta_r$, and by the snake property it follows that we have $\hat{\omega}_{\eta_s} = y$. Since we saw in the previous proof that $(\omega_{\eta_s}(t))_{0<t<\zeta_{\eta_s}}$ does not hit y, we get that $\tau_y^*(\omega_{\eta_s}) = \zeta_s$.

The truncation operation tr_y is a measurable mapping from $\mathcal{S}_x$ into $\mathcal{S}_x$. If $y \neq x$, it follows from the definition (and it was noticed in the preceding proof) that, if $\omega' = \mathrm{tr}_y(\omega)$ is the truncation of a snake trajectory $\omega \in \mathcal{S}_x$, the paths ω'_s stay in $[y, \infty)$ (if $y < x$) or in $(-\infty, y]$ (if $y > x$) and can only hit y at their lifetime.

The following lemma gives a simple continuity property of the truncation operations.

Lemma 3.2.7. *Let $\omega \in \mathcal{S}_0$ and $b < 0$. Suppose that*

$$\int_0^{\sigma(\omega)} \mathrm{d}s\, \mathbf{1}_{\{\tau_b(\omega_s) = \zeta_s(\omega)\}} = 0.$$

Then, for any sequence $(b_n)_{n \geq 1}$ such that $b_n \downarrow b$ as $n \to \infty$, we have $\mathrm{tr}_{b_n}(\omega) \longrightarrow \mathrm{tr}_b(\omega)$ in $\mathcal{S}$ as $n \to \infty$.

We omit the easy proof of this lemma. We conclude this subsection with a technical lemma that will be useful in the proof of one of our main results. Recall the notation $\underline{\mathrm{w}} = \min\{\mathrm{w}(t) : 0 \leq t \leq \zeta_{(\mathrm{w})}\}$ for $\mathrm{w} \in \mathcal{W}$.

Lemma 3.2.8. *Let $\omega \in \mathcal{S}$, and let ω' be a subtrajectory of ω associated with the interval $[a,b]$. Assume that $\omega' \in \mathcal{S}_0$ and, for every $n \geq 1$, let $\omega^{(n)}$ be a subtrajectory of ω associated with the interval $[a_n, b_n]$, such that $[a,b] \subset [a_n, b_n]$ for every $n \geq 1$ and $a_n \to a$, $b_n \to b$ as $n \to \infty$. Assume that the following properties hold:*

(i) $\omega_a(t) \geq 0$ for every $0 \leq t \leq \zeta_a(\omega)$;

(ii) for every $s \in (0, b-a)$, $\tau_0^*(\omega'_s) > 0$ and $\omega'_s(t) \geq 0$ for $0 \leq t \leq \tau_0^*(\omega_s) \wedge \zeta_{(\omega'_s)}$;

(iii) for every $s \in (0, b-a)$ such that $\zeta_s(\omega') > \tau_0^*(\omega'_s)$, we have $\underline{\omega}'_s < 0$.

Then, if $(\delta_n)_{n\geq 1}$ is any sequence of negative real numbers converging to 0, we have $\mathrm{tr}_{\delta_n}(\omega^{(n)}) \longrightarrow \mathrm{tr}_0(\omega')$ in $\mathcal{S}$ as $n \to \infty$.

Proof. The first step is to verify that $\omega^{(n)}$ converges to ω' in $\mathcal{S}$. To this end, let (ζ, f) be the tree-like path associated with ω, and notice that the tree-like path associated with $\omega^{(n)}$ is $(\zeta^{(n)}, f^{(n)})$, with $\zeta^{(n)}(r) = \zeta((a_n + r) \wedge b_n) - \zeta(a_n)$ and $f^{(n)}(r) = f((a_n + r) \wedge b_n)$. From the convergences $a_n \to a$, $b_n \to b$, it immediately follows that the pair $(\zeta^{(n)}, f^{(n)})$ converges to the tree-like path (ζ', f') associated with ω', and Proposition 3.2.4 implies that $\omega^{(n)}$ converges to ω'.

We also note that, for every $n \geq 1$, we have $f^{(n)}(0) = \omega_a(\zeta^{(n)}(0)) \geq 0$, since $p_\zeta(a_n)$ is an ancestor of $p_\zeta(a)$ and we use (i). By preceding remarks, we know that the paths of $\mathrm{tr}_{\delta_n}(\omega^{(n)})$ stay in $[\delta_n, \infty)$.

Set $\tilde{\omega}^{(n)} = \mathrm{tr}_{\delta_n}(\omega^{(n)})$ and $\tilde{\omega}' = \mathrm{tr}_0(\omega')$ to simplify notation. Then set, for every $s \geq 0$,

$$A_s^{(n)} := \int_0^s \mathrm{d}r\, \mathbf{1}_{\{\zeta^{(n)}(r) \leq \tau^*_{\delta_n}(\omega_r^{(n)})\}}\ , \quad A'_s := \int_0^s \mathrm{d}r\, \mathbf{1}_{\{\zeta'(r) \leq \tau_0^*(\omega'_r)\}},$$

and

$$\eta_s^{(n)} := \inf\{r \geq 0 : A_r^{(n)} > s\}\ , \quad \eta'_s := \inf\{r \geq 0 : A'_r > s\},$$

in such a way that $\tilde{\omega}_s^{(n)} = \omega^{(n)}_{\eta_s^{(n)}}$ and $\tilde{\omega}'_s = \omega'_{\eta'_s}$ by the definition of truncations. We observe that, for every $s \geq 0$, we have

$$A_s^{(n)} \underset{n\to\infty}{\longrightarrow} A'_s. \tag{3.2}$$

To see this, note that, for $r \in [a,b]$, the paths ω_r are the same as ω_a up to time $\zeta(a) = \zeta_a(\omega)$, and thus stay nonnegative on the time interval $[0, \zeta(a)]$ by assumption (i). From our definitions, it follows that the paths $\omega^{(n)}_{a-a_n+r}$, for $0 \leq r \leq b-a$, stay nonnegative up to time $\zeta(a) - \zeta(a_n) \geq 0$. Then, for $r \in [0, b-a]$, we have $\omega'_r(\cdot) = \omega^{(n)}_{a-a_n+r}(\zeta(a) - \zeta(a_n) + \cdot)$, and by (ii) we get that, if $\zeta'(r) \leq \tau_0^*(\omega'_r)$, the path $\omega^{(n)}_{a-a_n+r}$ does not hit $\delta_n < 0$ between times $\zeta(a) - \zeta(a_n)$ and $\zeta^{(n)}(a - a_n + r)$. Hence, we have, for every $r \in [0, b-a]$,

$$\mathbf{1}_{\{\zeta'(r) \leq \tau_0^*(\omega'_r)\}} \leq \mathbf{1}_{\{\zeta^{(n)}(a-a_n+r) \leq \tau^*_{\delta_n}(\omega^{(n)}_{a-a_n+r})\}}.$$

It follows that $A'_s \leq A^{(n)}_{a-a_n+s} \leq A^{(n)}_s + (a - a_n)$, which implies

$$\liminf_{n\to\infty} A_s^{(n)} \geq A'_s,$$

for every $s \geq 0$. Conversely, we claim that, for every $r \in (0, b-a)$,

$$\limsup_{n\to\infty} \mathbf{1}_{\{\zeta^{(n)}(r) \leq \tau^*_{\delta_n}(\omega_r^{(n)})\}} \leq \mathbf{1}_{\{\zeta'(r) \leq \tau_0^*(\omega'_r)\}}.$$

Indeed, if $\tau_0^*(\omega'_r) < \zeta'(r)$, then assumption (iii) implies that ω'_r takes negative values before its lifetime. From the convergence of $\omega_r^{(n)}$ to ω'_r, we get that we must have $\tau^*_{\delta_n}(\omega_r^{(n)}) < \zeta^{(n)}(r)$ for n large, proving our claim. The claim now gives

$$\limsup_{n\to\infty} A_s^{(n)} \leq A'_s,$$

completing the proof of (3.2). Notice that (3.2) also implies that $A^{(n)}_{b_n-a_n} \longrightarrow A'_{b-a}$, from which one gets that $\sigma(\tilde{\omega}^{(n)}) \longrightarrow \sigma(\tilde{\omega}')$, noting that $\sigma(\tilde{\omega}') = A'_{b-a}$ as a consequence of (ii) (if $0 < s < A'_{b-a}$, $\tilde{\omega}'_s = \omega'_{\eta'_s}$ is not a trivial path by (ii) and the fact that $0 < \eta'_s < b-a$).

It follows from (3.2) that we have $\eta_s^{(n)} \longrightarrow \eta'_s$, and consequently $\tilde{\omega}_s^{(n)} \longrightarrow \tilde{\omega}'_s$, as $n \to \infty$, for every $s \geq 0$ such that $\eta'_s = \eta'_{s-}$. To see that this implies the uniform convergence of $\tilde{\omega}^{(n)}$ toward $\tilde{\omega}'$, we argue by contradiction. Suppose that this uniform convergence does not hold, so that (modulo the extraction of a subsequence of $(\tilde{\omega}^{(n)})_{n\geq 1}$) we can find a sequence $(s_n)_{n\geq 1}$ and a real $\xi > 0$ such that, for every n,

$$d_{\mathcal{S}}(\tilde{\omega}^{(n)}_{s_n}, \tilde{\omega}'_{s_n}) > \xi. \tag{3.3}$$

Since both $\tilde{\omega}_r^{(n)}$ and $\tilde{\omega}'_r$ are constant (and equal to a trivial path) when $r \geq \sigma(\omega)$, we can assume that $s_n \in [0, \sigma(\omega)]$ for every n and then, modulo the extraction of a subsequence, that $s_n \longrightarrow s_\infty$ as $n \to \infty$. We must then have $\eta'_{s_\infty -} < \eta'_{s_\infty}$ because otherwise (3.2) would imply that $\eta^{(n)}_{s_n} \longrightarrow \eta_{s_\infty}$ and therefore $\tilde{\omega}^{(n)}_{s_n} \longrightarrow \tilde{\omega}'_{s_\infty}$, contradicting (3.3). We can also assume that $0 < s_\infty < \sigma(\tilde{\omega}')$, and therefore $0 < \eta'_{s_\infty} < b-a$, since it follows from assumption (ii) that η' is continuous at $\sigma(\tilde{\omega}') = A'_{b-a}$ (if $0 < s < b-a$, property (ii) and the snake property imply that the interval $[s, b-a]$ contains a set of positive Lebesgue measure of values of r such that $\tau_0^*(\omega(r)) = \infty$, and this is what we need to get the latter continuity property). Also notice that (ii) implies $\zeta'(r) > 0$ for $0 < r < b-a$ and consequently $\zeta'(\eta'_r) > 0$ for $0 < r < \sigma(\tilde{\omega}')$.

From (3.2), we get that any accumulation point of the sequence $(\eta^{(n)}_{s_n})_{n\geq 1}$ must lie in the interval $[\eta'_{s_\infty -}, \eta'_{s_\infty}]$. We claim that for any such accumulation point r we have $\omega'_r = \omega'_{\eta'_{s_\infty}}$. This implies that $\tilde{\omega}^{(n)}_{s_n} = \omega^{(n)}_{\eta^{(n)}_{s_n}}$ converges to $\omega'_{\eta'_{s_\infty}} = \tilde{\omega}'_{s_\infty}$ and contradicts (3.3). To verify our claim, let $r \in [\eta'_{s_\infty -}, \eta'_{s_\infty}]$ be an accumulation point of the sequence $(\eta^{(n)}_{s_n})_{n\geq 1}$. By property (3.1) in the proof of Proposition 3.2.6, we know that the path ω'_r coincides with $\omega'_{\eta'_{s_\infty}}$ up to $\zeta'(\eta'_{s_\infty}) = \tau_0^*(\omega'_{\eta'_{s_\infty}})$ (the last equality by the remark following Proposition 3.2.6). However, $\zeta'(r) > \zeta'(\eta'_{s_\infty})$ is impossible since assumption (iii) would imply that ω'_r takes negative values and cannot be an accumulation point of the sequence $\tilde{\omega}^{(n)}_{s_n}$ (because $\tilde{\omega}^{(n)}_{s_n}$ takes values in $[\delta_n, \infty)$ and δ_n tends to 0 as $n \to \infty$). Therefore we have $\zeta'(r) = \zeta'(\eta'_{s_\infty})$ meaning that $\omega'_r = \omega'_{\eta'_{s_\infty}}$ as desired. This completes the proof. □

3.2.3 The Brownian snake

In this section we discuss the (one-dimensional) Brownian snake excursion measures. We avoid defining the Brownian snake starting from a general initial value (which is briefly presented in the Introduction above) as this definition is not required in what follows, except in the proof of one technical lemma (Lemma 3.2.12) which the reader can skip at first reading.

Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy the assumptions of subsection 3.2.1 (including assumptions (i)–(iii) at the end of this subsection) and also assume that h is Hölder continuous with exponent δ for some $\delta > 0$. Let $(G^h_s)_{s\geq 0}$ be the centered real Gaussian process with covariance

$$\mathrm{cov}(G^h_s, G^h_t) = \min_{s\wedge t \leq r \leq s\vee t} h(r), \tag{3.4}$$

for every $s, t \geq 0$. We leave it as an exercise to verify that the right-hand side of (3.4) is a covariance function (see Lemma 4.1 in [49]). Note that we have then

$$E[(G_s^h - G_t^h)^2] = d_h(s,t). \tag{3.5}$$

An application of the classical Kolmogorov lemma shows that $(G_s^h)_{s\geq 0}$ has a continuous modification, which we consider from now on. Then property (3.5) entails that, for every fixed $0 \leq s \leq t$ such that $d_h(s,t) = 0$, we have $P(G_s^h = G_t^h) = 1$. A continuity argument, using the assumptions satisfied by h, then shows that, a.s., for every $0 \leq s \leq t$, the property $d_h(s,t) = 0$ implies $G_s^h = G_t^h$. This means that apart from a set of probability 0 which we may discard, the pair (h, G^h) is a (random) tree-like path in the sense of the preceding subsection.

The (one-dimensional) Brownian snake driven by h is the random snake trajectory $W^h = (W_s^h)_{s\geq 0}$ associated with the tree-like path (h, G^h). We write $\mathbf{P}_h(\mathrm{d}\omega)$ for the law of W^h on the space $\mathcal{S}_0$.

We next randomize h: We let $\mathbf{n}(\mathrm{d}h)$ stand for Itô's excursion measure of positive excursions of linear Brownian motion (see e.g. [63, Chapter XII]) normalized so that, for every $\varepsilon > 0$,

$$\mathbf{n}\Big(\max_{s\geq 0} h(s) > \varepsilon\Big) = \frac{1}{2\varepsilon}.$$

Notice that $\mathbf{n}$ is supported on functions h that satisfy the assumptions required above to define W^h and the probability measure $\mathbf{P}_h(\mathrm{d}\omega)$. The Brownian snake excursion measure $\mathbb{N}_0$ is then the σ-finite measure on $\mathcal{S}_0$ defined by

$$\mathbb{N}_0(\mathrm{d}\omega) = \int \mathbf{n}(\mathrm{d}h)\, \mathbf{P}_h(\mathrm{d}\omega).$$

In other words, the "lifetime process" $(\zeta_s)_{s\geq 0}$ is distributed under $\mathbb{N}_0(\mathrm{d}\omega)$ according to Itô's measure $\mathbf{n}(\mathrm{d}h)$, and, conditionally on $(\zeta_s)_{s\geq 0}$, $(W_s)_{s\geq 0}$ is distributed as the Brownian snake driven by $(\zeta_s)_{s\geq 0}$. The reader will easily check that the preceding definition of $\mathbb{N}_0$ is consistent with the slightly different presentation given in the Introduction above (see [44] for more details about the Brownian snake). For every $x \in \mathbb{R}$, we also define $\mathbb{N}_x$ as the measure on $\mathcal{S}_x$ which is the image of $\mathbb{N}_0$ under the translation κ_x.

Let us recall the first-moment formula for the Brownian snake [44, Section IV.2]. For every nonnegative measurable function ϕ on $\mathcal{W}$,

$$\mathbb{N}_x\Big(\int_0^\sigma \mathrm{d}s\, \phi(W_s)\Big) = \mathbb{E}_x\Big[\int_0^\infty \mathrm{d}t\, \phi\big((B_r)_{0\leq r\leq t}\big)\Big], \tag{3.6}$$

where $B = (B_r)_{r\geq 0}$ stands for a linear Brownian motion starting from x under the probability measure $\mathbb{P}_x$.

We define the range $\mathcal{R}$ by

$$\mathcal{R} := \{\hat{W}_s : s \geq 0\} = \{W_s(t) : s \geq 0, 0 \leq t \leq \zeta_s\},$$

and we set

$$W_* := \min \mathcal{R}.$$

Then, if $x, y \in \mathbb{R}$ and $y < x$, we have

$$\mathbb{N}_x(W_* \leq y) = \frac{3}{2(x-y)^2}. \tag{3.7}$$

See e.g. [44, Section VI.1].

3.2.4 Exit measures and the special Markov property

Let U be a nonempty open interval of $\mathbb{R}$, such that $U \neq \mathbb{R}$. For any $\mathrm{w} \in \mathcal{W}$, set

$$\tau^U(\mathrm{w}) := \inf\{t \in [0, \zeta_{(\mathrm{w})}] : \mathrm{w}(t) \notin U\}.$$

If $x \in U$, the limit

$$\langle \mathcal{Z}^U, \phi \rangle = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\sigma \mathrm{d}s \, \mathbf{1}_{\{\tau^U(W_s) < \zeta_s < \tau^U(W_s) + \varepsilon\}} \, \phi(W_s(\tau^U(W_s))) \tag{3.8}$$

exists $\mathbb{N}_x$ a.e. for any function ϕ on ∂U and defines a finite random measure $\mathcal{Z}^U$ supported on ∂U (see [44, Chapter V]). Notice that here ∂U has at most two points, but the preceding definition holds in the same form for the Brownian snake in higher dimensions with an arbitrary open set U. Informally, the measure $\mathcal{Z}^U$ "counts" the exit points of the paths W_s from U, for those values of s such that W_s exits U. In particular, $\mathcal{Z}^U = 0$ if none of the paths W_s exits U.

Exit measures are needed to state the so-called special Markov property, which plays a key role in this work. Before stating this property, we introduce the excursions outside U of a snake trajectory. We fix $x \in U$ and we let $\omega \in \mathcal{S}_x$. We observe that the set

$$\{s \geq 0 : \tau^U(\omega_s) < \zeta_s\}$$

is open and can therefore be written as a union of disjoint open intervals (a_i, b_i), $i \in I$, where I may be empty. From the fact that ω is a snake trajectory, it is not hard to verify that we must have $p_\zeta(a_i) = p_\zeta(b_i)$ for every $i \in I$, where we recall that p_ζ is the canonical projection from $\mathbb{R}_+$ onto the tree $\mathcal{T}_\zeta$ coded by $(\zeta_s(\omega))_{s \geq 0}$. Furthermore the path $\omega_{a_i} = \omega_{b_i}$ exits U exactly at its lifetime $\zeta_{a_i} = \zeta_{b_i}$. We can then define the excursion ω_i, for every $i \in I$, as the subtrajectory of ω associated with the interval $[a_i, b_i]$ (equivalently $W_s(\omega_i)$ is the finite path $(\omega_{(a_i+s)\wedge b_i}(\zeta_{a_i} + t))_{0 \leq t \leq \zeta^i(s)}$ with lifetime $\zeta^i(s) = \zeta_{(a_i+s)\wedge b_i} - \zeta_{a_i}$, for every $s \geq 0$). The ω_i's are the "excursions" of the snake trajectory ω outside U – the word "outside" is a little misleading here, because although these excursions start from ∂U, they will typically come back inside U. We define the point measure of excursions of ω outside U by

$$\mathcal{P}^U(\omega) := \sum_{i \in I} \delta_{\omega_i}.$$

We also need to define the σ-field on $\mathcal{S}_x$ containing the information given by the paths ω_s before they exit U. To this end we generalize a little the definition of truncations in subsection 3.2.2. If $\omega \in \mathcal{S}_x$, we set

$$\mathrm{tr}^U(\omega)_s := \omega_{\eta^U_s}$$

where

$$\eta^U_s := \inf\{r \geq 0 : \int_0^r \mathrm{d}t \, \mathbf{1}_{\{\zeta_t(\omega) \leq \tau^U(\omega_t)\}} > s\}.$$

Just as in Proposition 3.2.6, we can verify that this defines a measurable mapping from $\mathcal{S}_x$ into $\mathcal{S}_x$. We define the σ-field $\mathcal{E}^U_x$ on $\mathcal{S}_x$ as the σ-field generated by this mapping and completed by the measurable sets of $\mathcal{S}_x$ of $\mathbb{N}_x$-measure 0.

We can now state the special Markov property.

Proposition 3.2.9. *Let $x \in U$. The random measure $\mathcal{Z}^U$ is $\mathcal{E}_x^U$-measurable. Furthermore, under the probability measure $\mathbb{N}_x(\cdot \mid \mathcal{R} \cap U^c \neq \varnothing)$, conditionally on $\mathcal{E}_x^U$, the point measure $\mathcal{P}^U$ is Poisson with intensity*

$$\int \mathcal{Z}^U(\mathrm{d}y)\, \mathbb{N}_y(\cdot).$$

See [46, Theorem 2.4] for a proof in a much more general setting. Note that, on the event $\{\mathcal{R} \cap U^c = \varnothing\}$, there are no excursions outside U, and this is the reason why we restrict our attention to the event $\{\mathcal{R} \cap U^c \neq \varnothing\}$, which has finite $\mathbb{N}_x$-measure by (3.7).

3.2.5 The exit measure process

We now specialize the discussion of the previous subsection to the case $U = (y, \infty)$ and $x > y$. The exit measure $\mathcal{Z}^{(y,\infty)}$ is then a random multiple of the Dirac mass at y, and is determined by its total mass, which will be denoted by $\mathcal{Z}_y = \langle \mathcal{Z}^{(y,\infty)}, 1\rangle$. We have

$$\{\mathcal{Z}_y > 0\} = \{W_* < y\} = \{W_* \leq y\}\ , \quad \mathbb{N}_x \text{ a.e.}$$

Note that the identity $\{W_* < y\} = \{W_* \leq y\}$, $\mathbb{N}_x$ a.e., follows from the fact that the right-hand side of (3.7) is a continuous function of y. The fact that $\{\mathcal{Z}_y > 0\} = \{W_* < y\}$, $\mathbb{N}_x$ a.e., can then be deduced from the special Markov property.

The Laplace transform of $\mathcal{Z}_y$ under $\mathbb{N}_x$ can be computed from the connections between exit measures and semilinear partial differential equations [44, Chapter V]. For every $\lambda > 0$,

$$\mathbb{N}_x(1 - \exp(-\lambda \mathcal{Z}_y)) = \Big(\lambda^{-1/2} + \sqrt{\frac{2}{3}}\,(x-y)\Big)^{-2}. \tag{3.9}$$

See formula (6) in [25] for a brief justification. Note that letting $\lambda \to \infty$ in (3.9) is consistent with (3.7). A consequence of (3.9) is the fact that

$$\mathbb{N}_x(\mathcal{Z}_y) = 1. \tag{3.10}$$

Let us discuss Markovian properties of the process of exit measures. If $y' < y < x$, an application of the special Markov property combined with formula (3.9) gives on the event $\{W_* \leq y\}$,

$$\mathbb{N}_x\Big(\exp -\mathcal{Z}_{y'} \,\Big|\, \mathcal{E}_x^{(y,\infty)}\Big) = \exp\Big(-\mathcal{Z}_y\, \mathbb{N}_y(1 - \exp(-\lambda \mathcal{Z}_{y'}))\Big) = \exp -\mathcal{Z}_y\Big(\lambda^{-1/2} + \sqrt{\frac{2}{3}}\,(y-y')\Big)^{-2}.$$

It follows that the process $(\mathcal{Z}_{x-a})_{a>0}$ is Markovian under $\mathbb{N}_x$, with the transition kernels of the continuous-state branching process with branching mechanism $\psi(\lambda) = \sqrt{8/3}\ \lambda^{3/2}$ (see e.g. [25, Section 2.1] for the definition and properties of this process). Although $\mathbb{N}_x$ is an infinite measure, the previous statement makes sense by arguing on the event $\{W_* \leq x - \delta\}$, which has finite $\mathbb{N}_x$-measure for any $\delta > 0$, and considering $(\mathcal{Z}_{x-\delta-a})_{a\geq 0}$.

We will use an approximation of $\mathcal{Z}_y$ by $\mathcal{E}_x^{(y,\infty)}$-measurable random variables (notice that this is not the case for (3.8)). Recall our notation $\tau_y(\mathrm{w}) := \inf\{t \in [0, \zeta_{(\mathrm{w})}] : \mathrm{w}(t) = y\}$ for $\mathrm{w} \in \mathcal{W}$.

Lemma 3.2.10. *We have*

$$\varepsilon^{-2} \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\zeta_s \leq \tau_y(W_s), \hat{W}_s < y+\varepsilon\}} \underset{\varepsilon \to 0}{\longrightarrow} \mathcal{Z}_y$$

where the convergence holds in probability under $\mathbb{N}_x(\cdot \mid W_* \leq y)$.

Proof. This follows from arguments similar to the proof of Proposition 1.1 in [25, Section 4.1], and we only sketch the proof. For every $\varepsilon > 0$, set

$$\Lambda_\varepsilon = \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\zeta_s \le \tau_y(W_s), \hat{W}_s < y+\varepsilon\}}.$$

If $\varepsilon \in (0, x-y)$, the special Markov property applied to the domain $(y+\varepsilon, \infty)$ shows that the conditional distribution of Λ_ε, under $\mathbb{N}_x(\cdot \mid W_* \le y+\varepsilon)$ and knowing $\mathcal{E}^{(y+\varepsilon,\infty)}$, is the law of $S_\varepsilon(\mathcal{Z}_{y+\varepsilon})$, where $(S_\varepsilon(t))_{t\ge 0}$ is a subordinator whose Lévy measure is the "law" of Λ_ε under $\mathbb{N}_{y+\varepsilon}$, and S_ε is assumed to be independent of Λ_ε. The first-moment formula for the Brownian snake (3.6) gives $\mathbb{N}_{y+\varepsilon}(\Lambda_\varepsilon) = \varepsilon^2$, so that $S_\varepsilon(t)$ has mean $\varepsilon^2 t$. On the other hand, scaling arguments entail that $(S_\varepsilon(t))_{t\ge 0}$ has the same distribution as $(\varepsilon^4 S_1(\varepsilon^{-2}t))_{t\ge 0}$. Hence, under $\mathbb{N}_x(\cdot \mid W_* \le y+\varepsilon)$ and conditionally on $\mathcal{E}^{(y+\varepsilon,\infty)}$, $\varepsilon^{-2}\Lambda_\varepsilon$ has the law of $\varepsilon^2 S_1(\varepsilon^{-2}\mathcal{Z}_{y+\varepsilon})$, and the latter random variable is close in probability to $\mathcal{Z}_{y+\varepsilon}$ by the law of large numbers ($t^{-1}S_1(t)$ converges in probability to 1 as $t\to\infty$). The result of the lemma follows since $\mathcal{Z}_{y+\varepsilon}$ converges to $\mathcal{Z}_y$ in probability when $\varepsilon \to 0$. $\square$

We note that the quantities $\int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\zeta_s \le \tau_y(W_s), \hat{W}_s < y+\varepsilon\}}$ are functions of the truncation $\mathrm{tr}_y(\omega)$, and therefore $\mathcal{E}_x^{(y,\infty)}$-measurable. As a consequence of Lemma 3.2.10, we can fix a sequence $(\alpha_n)_{n\ge 1}$ of positive reals converging to 0 such that

$$\mathcal{Z}_y = \lim_{n\to\infty} \alpha_n^{-2} \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\zeta_s \le \tau_y(W_s), \hat{W}_s < y+\alpha_n\}}\,, \qquad \mathbb{N}_x \text{ a.e.} \tag{3.11}$$

and we can even choose the sequence $(\alpha_n)_{n\ge 1}$ independently of x and y such that $y < x$ (observe that if (3.11) holds for $y = x-\delta$, then an application of the special Markov property shows that it holds for every $y \in (-\infty, x-\delta]$). It will be convenient to define $\mathcal{Z}_y(\omega)$ for every $\omega \in S_x$, by setting

$$\mathcal{Z}_y(\omega) = \liminf_{n\to\infty} \alpha_n^{-2} \int_0^{\sigma(\omega)} \mathrm{d}s\, \mathbf{1}_{\{\zeta_s(\omega) \le \tau_y(W_s(\omega)), \hat{W}_s(\omega) < y+\alpha_n\}}.$$

With this convention, we have $\mathcal{Z}_y(\omega) = \mathcal{Z}_y(\mathrm{tr}_y(\omega))$.

In much of what follows, we will argue under the measure $\mathbb{N}_0$, and we simply write $\mathcal{E}^{(y,\infty)}$ instead of $\mathcal{E}_0^{(y,\infty)}$, for every $y < 0$. For $\omega \in \mathcal{S}_0$, we use the notation

$$Z_a(\omega) = \mathcal{Z}_{-a}(\omega)$$

for every $a > 0$. Because continuous-state branching processes are Feller processes, we know that the process $(Z_a)_{a>0}$ has a càdlàg modification under $\mathbb{N}_0$, and we will always consider this modification. We call $(Z_a)_{a>0}$ the exit measure process.

We will need some bounds on the moments of Z_a. By (3.10), we already know that $\mathbb{N}_0(Z_a) = 1$ for every $a > 0$. Moreover, an application of the special Markov property shows that the process $(Z_{\delta+a})_{a\ge 0}$ is a martingale under $\mathbb{N}_0(\cdot \mid W_* \le -\delta)$, for every $\delta > 0$ (this also follows from the fact that $\psi(\lambda) = \sqrt{8/3}\,\lambda^{3/2}$ is a critical branching mechanism).

Lemma 3.2.11. *Let $p \in (1, 3/2)$. For every $a > 0$, the quantities $\mathbb{N}_0((Z_b)^p)$, $0 < b \le a$, are bounded.*

Proof. Write $\mathbb{N}_0^{(a)} := \mathbb{N}_0(\cdot \mid W_* \leq -a)$ to simplify notation. As a consequence of (3.9) and (3.7), we get that, for every $\lambda > 0$,

$$\mathbb{N}_0^{(a)}\Big(e^{-\lambda Z_a}\Big) = 1 - \Big(1 + a^{-1}\sqrt{\frac{3}{2\lambda}}\Big)^{-2},$$

and we have also $\mathbb{N}_0^{(a)}(Z_a) = 2a^2/3$. From a Taylor expansion, we get

$$\mathbb{N}_0^{(a)}\Big(e^{-\lambda Z_a}\Big) - (1 - \lambda \mathbb{N}_0^{(a)}(Z_a)) = 2\Big(\frac{2}{3}\Big)^{3/2} a^3\,\lambda^{3/2} + o(\lambda^{3/2}),$$

as $\lambda \to 0$. By [14, Theorem 8.1.6], this implies the existence of a constant C such that $\mathbb{N}_0^{(a)}(Z_a > x) \leq C\,x^{-3/2}$ for every $x > 0$. Consequently, $\mathbb{N}_0^{(a)}((Z_a)^p) < \infty$ if $1 < p < 3/2$.

Finally, if $b \in (0, a)$, we get by using the martingale property of the exit measure process,

$$\mathbb{N}_0((Z_b)^p) = \frac{3}{2b^2}\,\mathbb{N}_0^{(b)}((Z_b)^p) \leq \frac{3}{2b^2}\,\mathbb{N}_0^{(b)}((Z_a)^p) = \mathbb{N}_0((Z_a)^p) < \infty.$$

□

3.2.6 A technical lemma

We finally give a technical lemma concerning local minima of the process $\hat{W}$.

Lemma 3.2.12. $\mathbb{N}_0$ *a.e., there exists no value of* $s \in (0, \sigma)$ *such that:*

(i) *s is a time of local minimum of $\hat{W}$, in the sense that there exists $\varepsilon > 0$ such that $\hat{W}_r \geq \hat{W}_s$ for every $r \in (s - \varepsilon, s + \varepsilon)$.*

(ii) *$\hat{W}_s = \underline{W}_s$ and there exists $t \in (0, \zeta_s)$ such that $W_s(t) = \underline{W}_s$.*

Proof. The proof uses more involved properties of the Brownian snake which we have not recalled but for which we refer the reader to [44]. We start by observing that, for every reals $y < x$, we have $\mathbb{N}_x$ a.e.

$$\inf\{s \geq 0 : \hat{W}_s < y\} = \inf\{s \geq 0 : \hat{W}_s \leq y\}. \tag{3.12}$$

In other words, when the Brownian snake hits y, it immediately hits values strictly smaller than y. See the proof of Theorem VI.9 in [44] for an argument in a more general setting.

Then, fix $\mathrm{w} \in \mathcal{W}_0$ and let $(W'_s)_{s\geq 0}$ be a Brownian snake that starts from w under the probability measure $\mathbb{P}_{\mathrm{w}}$ (we write W'_s and not W_s because $\mathbb{P}_{\mathrm{w}}$ is not defined on the space $\mathcal{S}$ of snake trajectories). We let $(\zeta'_s)_{s\geq 0}$ be the lifetime process of $(W'_s)_{s\geq 0}$. Suppose that there is a unique time $t_0 \in (0, \zeta_{(\mathrm{w})})$ such that $\mathrm{w}(t_0) = \underline{\mathrm{w}}$, and introduce the stopping time

$$\tau := \inf\{s \geq 0 : \zeta'_s \leq t_0\}.$$

Notice that the path W'_τ is equal to the restriction of w to $[0, t_0]$, and thus $\hat{W}'_\tau = \mathrm{w}(t_0) = \underline{\mathrm{w}}$. We then claim that, $\mathbb{P}_{\mathrm{w}}$ a.s. on the event where $\inf\{s > 0 : \hat{W}'_s \leq \underline{\mathrm{w}}\} < \tau$, we have

$$\inf\{s > 0 : \hat{W}'_s \leq \underline{\mathrm{w}}\} = \inf\{s > 0 : \hat{W}'_s < \underline{\mathrm{w}}\}.$$

This follows by using the subtree decomposition of the Brownian snake started at w (see [44, Lemma V.5]) together with property (3.12) above.

We can now combine the previous observations with the Markov property of the Brownian snake under $\mathbb{N}_0$. We obtain that $\mathbb{N}_0$ a.e. for every rational $r \in (0, \sigma)$ such that $t \mapsto W_r(t)$ attains its minimum at a (necessarily unique) time $t_0 \in (0, \zeta_r)$, the property

$$\inf\{s > r : \hat{W}_s \leq \underline{W}_r\} < \inf\{s \geq r : \zeta_s \leq t_0\}$$

implies

$$\inf\{s > r : \hat{W}_s < \underline{W}_r\} = \inf\{s > r : \hat{W}_s \leq \underline{W}_r\}. \tag{3.13}$$

Let show that this implies the statement of the lemma. We argue by contradiction, assuming that there is a value $s_0 \in (0, \sigma)$ such that properties (i) and (ii) hold for $s = s_0$. Write t_0 for the (unique) time in $(0, \zeta_{s_0})$ such that $W_{s_0}(t_0) = \underline{W}_{s_0}$ and choose $\delta > 0$ such that $t_0 < \zeta_{s_0} - \delta$. Then, using property (i) for $s = s_0$ and the properties of the Brownian snake, we can find a rational $r < s_0$ sufficiently close to s_0 so that, for some $\eta > 0$,

(a) $\hat{W}_s \geq \hat{W}_{s_0}$ for every $s \in [r, s_0 + \eta]$;

(b) $\zeta_r + \delta/2 > \zeta_s > \zeta_r - \delta/2$ for every $s \in [r, s_0]$.

We note that W_r coincides with W_{s_0} at least up to time $\zeta_r - \delta/2 > \zeta_{s_0} - \delta > t_0$. In particular t_0 is also the unique time of the minimum of $t \mapsto W_r(t)$ on $(0, \zeta_r)$, and $\underline{W}_r = \underline{W}_{s_0} = \hat{W}_{s_0}$ (it already follows from property (a) that $\underline{W}_r \geq \hat{W}_{s_0}$). Property (b) then gives

$$\inf\{s > r : \hat{W}_s \leq \underline{W}_r\} \leq s_0 < \inf\{s \geq r : \zeta_s \leq t_0\}.$$

This allows us to apply (3.13) and to get

$$\inf\{s > r : \hat{W}_s < \underline{W}_r\} = \inf\{r' > r : \hat{W}_s \leq \underline{W}_r\} \leq s_0.$$

Since $\underline{W}_r = \hat{W}_{s_0}$, this contradicts property (a) above, and this contradiction completes the proof. $\square$

3.3 Construction of the excursion measure above the minimum

The main goal of this section is to construct the positive excursion measure $\mathbb{N}_0^*$. For this construction, we will be arguing under the excursion measure $\mathbb{N}_0$. Recall the notation $\mathcal{T}_\zeta$ for the random real tree coded by $(\zeta_s)_{s\geq 0}$, and $\mathrm{Sk}(\mathcal{T}_\zeta)$ for the skeleton of $\mathcal{T}_\zeta$. If $u \in \mathcal{T}_\zeta$ and $s \geq 0$ is such that $p_\zeta(s) = u$, we already noticed that W_s does not depend on the choice of s, and it will be convenient to write $V_u = \hat{W}_s$. Recall that V_u is interpreted as the label or spatial position of u.

Definition 3.3.1. *A vertex $u \in \mathcal{T}_\zeta$ is an excursion debut above the minimum if the following three properties hold:*

1. $u \in \mathrm{Sk}(\mathcal{T}_\zeta)$;
2. $V_u = \min\{V_v : v \in [\![\rho, u]\!]\}$;

3. u has a strict descendant w such that such that $V_v > V_u$ for all $v \in]\!]u, w[\![$.

We write D for the set of all excursion debuts above the minimum. If $u \in D$, V_u is called the level of the excursion debut u.

In what follows, except in Section 3.8, we will be interested only in excursions above the minimum, and for this reason we will say excursion debut instead of excursion debut above the minimum. By definition, excursion debuts belong to the skeleton of $\mathcal{T}_\zeta$. Clearly the root ρ is not an excursion debut (it is easy to see that property (3) fails for $u = \rho$) and we have $V_u < 0$ for every $u \in D$. Furthermore, the quantities $V_u, u \in D$ are pairwise distinct, as a consequence of the fact that local minima of Brownian paths are distinct (this fact implies that two local minima of labels that correspond to disjoint segments of the tree $\mathcal{T}_\zeta$ must be distinct).

Lemma 3.3.2. $\mathbb{N}_0$ *a.e., no branching point is an excursion debut.*

Proof. Any branching point can be represented as $p_\zeta(r)$, where $r \in (s,t)$ and $\zeta_r = \min\{\zeta_{r'} : s \leq r' \leq t\}$, for rationals s and t such that $0 < s < t < \sigma$. Then, for any strict descendant w of $p_\zeta(r)$, the historical path of w coincides either with W_s or with W_t, up to a time (strictly) greater than ζ_r. Since, conditionally on the lifetime process ζ, W_s is just a Brownian path over the time interval $[0, \zeta_s]$, it must take values smaller than $W_s(\zeta_r)$ immediately after time ζ_r, a.s., and the same holds for W_t. We conclude that $p_\zeta(r)$ is a.s. not an excursion debut, and by varying s and t we get the desired result outside a countable union of negligible sets. $\square$

Let u be an excursion debut. We set

$$C_u = \{w \in \mathcal{T}_\zeta : u \prec w \text{ and } V_v > V_u, \ \forall v \in]\!]u, w[\![\},$$

where we recall that the notation $v \prec w$ means that v is an ancestor of w. Note that $u \in C_u$ and that saying that u is an excursion debut implies that $C_u \neq \{u\}$. We have clearly $V_w \geq V_u$ for every $w \in C_u$. Also, if $w \in C_u$, then $w' \in C_u$ for every $w' \in [\![u, w[\![$.

Lemma 3.3.3. *For every $u \in D$, the set C_u is a closed subset of $\mathcal{T}_\zeta$ and its interior is*

$$\mathrm{Int}(C_u) = \{w \in C_u : V_w > V_u\}. \tag{3.14}$$

Proof. The fact that C_u is closed is easy: If (w_n) is a sequence in C_u that converges to w for the metric of $\mathcal{T}_\zeta$, then we have $u \prec w$ and the "interval" $]\!]u, w[\![$ is contained in the union of the intervals $]\!]u, w_n[\![$.

To verify (3.14), first note that the set $\{w \in C_u : V_w > V_u\}$ is open (if w belongs to this set and if w' is sufficiently close to w, then w' is still a descendant of u and $V_v > V_u$ for all $v \in]\!]u, w']\!]$).

We also need to check that, if $w \in C_u$ and $V_w = V_u$, then w does not belong to the interior of C_u. Consider first the case $w = u$. Letting s_1 be the first time such that $p_\zeta(s) = u$, the fact that u belongs to the interior of C_u would imply that $\hat{W}_s \geq \hat{W}_{s_1} = V_u$ for all $s \geq s_1$ sufficiently close to s_1. But then s_1 would a point of (right) increase for both ζ and $\hat{W}$, and by Lemma 2.2 in [47] we know that this cannot occur. Suppose then that $w \in C_u$, $V_w = V_u$ and $w \neq u$. Let $s \in (0, \sigma)$ such that $p_\zeta(s) = w$. Then property (ii) of Lemma 3.2.12 holds, and thus property (i) of the same lemma cannot hold. This shows that, for any neighborhood $\mathcal{N}$ of w we can find $w' \in \mathcal{N}$ such that $V_{w'} < V_u$ and therefore $w' \notin C_u$. $\square$

Proposition 3.3.4. *The sets* $\mathrm{Int}(C_u)$*, when* u *varies in* D*, are exactly the connected components of the open set* $\{w \in \mathcal{T}_\zeta : V_w > \min\{V_v : v \in [\![\rho, w]\!]\}\}$.

Proof. If $w \in \mathcal{T}_\zeta$ is such that $V_w > \min\{V_v : v \in [\![\rho, w]\!]\}$, then $w \in \mathrm{Int}(C_u)$, where u is the last ancestor of w such that $V_u = \min\{V_v : v \in [\![\rho, w]\!]\}$. This shows that $\{w \in \mathcal{T}_\zeta : V_w > \min\{V_v : v \in [\![\rho, w]\!]\}\}$ is the union of all sets $\mathrm{Int}(C_u)$, when u varies in D. Then, if $u \in D$ and w and w' are two vertices in $\mathrm{Int}(C_u)$, their last common ancestor also belongs to $\mathrm{Int}(C_u)$ (because u is not a branching point, by Lemma 3.3.2), and the whole interval $[\![w, w']\!]$ is contained in $\mathrm{Int}(C_u)$. It follows that, for every $u \in D$, the set $\mathrm{Int}(C_u)$ is connected. Finally, if u and u' are two distinct vertices in D, the sets $\mathrm{Int}(C_u)$ and $\mathrm{Int}(C_{u'})$ are disjoint. To see this, argue by contradiction and suppose that there exists $v \in \mathrm{Int}(C_u) \cap \mathrm{Int}(C_{u'})$, then u and u' are both ancestors of v, hence u is an ancestor of u' (or u' is an ancestor of u). However, the properties $u \prec u' \prec v$ and $v \in \mathrm{Int}(C_u)$ imply that $V_{u'} > V_u$, which contradicts property (2) in the definition of an excursion debut. □

Remark. A minor modification of the end of the proof shows in fact that the sets C_u, $u \in D$ are pairwise disjoint, which is slightly stronger.

The last proposition implies that the set D is countable, which can also be seen directly.

Definition 3.3.5. *If* u *is an excursion debut, we set*

$$M_u := \sup\{V_v - V_u : v \in C_u\} > 0$$

and we call M_u *the height of the excursion debut* u*. For every* $\delta > 0$*, we set* $D_\delta := \{u \in D : M_u > \delta\}$.

Lemma 3.3.6. *Let* $\delta > 0$*. The set* D_δ *is finite* $\mathbb{N}_0$ *a.e.*

Proof. By a uniform continuity argument, there exists a (random) $\eta > 0$ such that, for every $v, v' \in \mathcal{T}_\zeta$, the condition $d_\zeta(v, v') \le \eta$ implies $|V_v - V_{v'}| \le \delta$. Then let $u \in D_\delta$, and let $v \in C_u$ such that $V_v - V_u > \delta$. We claim that the ball of radius $\eta/2$ centered at v in $\mathcal{T}_\zeta$, which we denote by $B_{d_\zeta}(v, \eta/2)$, is contained in $\mathrm{Int}(C_u)$. If the claim holds, the result of the lemma follows since the sets $\mathrm{Int}(C_u)$ are disjoint when u varies (Proposition 3.3.4), and there can only be finitely many values of v such that the balls $B_{d_\zeta}(v, \eta/2)$ are disjoint.

To verify our claim, we first note that we must have $d_\zeta(u, v) > \eta$ by our choice of η, and it follows that the ball $B_{d_\zeta}(v, \eta/2)$ is contained in the set of descendants of u. Next, if $v' \in B_{d_\zeta}(v, \eta/2)$, we have, for every $w \in [\![v, v']\!]$, $V_w \ge V_v - \delta > V_u$, showing that $v' \in \mathrm{Int}(C_u)$ since $[\![u, v']\!] \subset [\![u, v]\!] \cup [\![v, v']\!]$. This gives our claim and completes the proof. □

Let u be an excursion debut. Since $u \in \mathrm{Sk}(\mathcal{T}_\zeta)$ and u is not a branching point, there are two uniquely defined times $0 < s_1 < s_2 < \sigma$ such that $p_\zeta(s_1) = p_\zeta(s_2) = u$. Note that $\hat{W}_{s_1} = \hat{W}_{s_2} = V_u$ and $\zeta_{s_1} = \zeta_{s_2} = d_\zeta(\rho, u)$. We then define a random snake trajectory $W^{(u)} \in \mathcal{S}_0$ as the image under the translation κ_{-V_u} of the subtrajectory of ω associated with the interval $[s_1, s_2]$ (recall that the latter subtrajectory corresponds to the spatial displacements of the descendants of u). Note that $W^{(u)}$ has duration $\sigma(W^{(u)}) = s_2 - s_1$. Alternatively, the tree-like path corresponding to $W^{(u)}$ is $(\zeta_{(s_1+s)\wedge s_2} - \zeta_{s_1}, \hat{W}^{(u)}_{(s_1+s)\wedge s_2} - V_u)_{s \ge 0}$. By the definition of D, each of the paths $W^{(u)}_s$, for $0 < s < s_2 - s_1$, stays strictly above 0 during a small interval $(0, \delta)$, for some $\delta > 0$. We are in fact not interested in the behavior of these paths after they return to 0 (if they do) and, for this reason, we introduce the truncation of $W^{(u)}$ at 0,

$$\tilde{W}^{(u)} := \mathrm{tr}_0(W^{(u)}),$$

with the notation introduced in subsection 3.2.2. We also write $\tilde{\zeta}_s^{(u)}$ for the lifetime of $\tilde{W}_s^{(u)}$, for every $s \geq 0$. For every $s \in (0, \sigma(\tilde{W}^{(u)}))$, the path $\tilde{W}_s^{(u)}$ starts from 0, stays positive during the interval $(0, \tilde{\zeta}_s^{(u)})$ and may or may not return to 0 at time $\tilde{\zeta}_s^{(u)}$.

It follows from our definitions that the paths $\tilde{W}_s^{(u)}$, $0 \leq s \leq \sigma(\tilde{W}^{(u)})$, correspond to the historical paths after time $d_\zeta(\rho, u)$ of all vertices $v \in C_u$, provided these paths are shifted by $-V_u$ so that they start from 0. In particular, $M(\tilde{W}^{(u)}) = M_u$ is the height of the excursion debut u. We sometimes call $\tilde{W}^{(u)}$ the excursion above the minimum starting from u.

Before stating the main theorem of this section, we introduce one more notation. On the canonical space $\mathcal{S}$, we let $\tilde{W} = \mathrm{tr}_0(W)$ stand for the truncation at 0 of the canonical process $(W_s)_{s\geq 0}$.

Theorem 3.3.7. *There exists a σ-finite measure denoted by $\mathbb{N}_0^*$ on the space $\mathcal{S}$, which is supported on $\mathcal{S}_0$, such that for every nonnegative measurable function G on $\mathcal{S}$ and every nonnegative measurable function g on $\mathbb{R}$, we have*

$$\mathbb{N}_0\left(\sum_{u\in D} g(V_u)G(\tilde{W}^{(u)})\right) = \left(\int_{-\infty}^0 g(\ell)d\ell\right)\mathbb{N}_0^*(G). \tag{3.15}$$

The measure $\mathbb{N}_0^*$ gives finite mass to the set $\mathcal{S}^{(\delta)} := \{\omega \in \mathcal{S} : \|\omega\| > \delta\}$, for every $\delta > 0$. Moreover, if G is bounded and continuous on $\mathcal{S}$ and there exists $\delta > 0$ such that G vanishes on $\mathcal{S} \setminus \mathcal{S}^{(\delta)}$, we have

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\,\mathbb{N}_\varepsilon\Big(G(\tilde{W})\Big) = \mathbb{N}_0^*(G). \tag{3.16}$$

The proof of Theorem 3.3.7 relies on an important technical lemma, which we state after introducing some notation. We consider a fixed sequence $(\varepsilon_n)_{n\geq 1}$ of positive real numbers converging to 0. We let ε be an element of this sequence, then for every $\omega \in \mathcal{S}_0$ and for every integer $k \geq 1$, we let $\mathcal{N}_k^\varepsilon(\omega)$ be the point measure of excursions of ω outside $(-k\varepsilon, \infty)$, and we write

$$\mathcal{N}_k^\varepsilon(\omega) = \sum_{i\in I_k^\varepsilon} \delta_{\omega_i^{k,\varepsilon}}.$$

By construction, for every $i \in I_k^\varepsilon$, $\omega_i^{k,\varepsilon}$ is a subtrajectory of ω, and we write $[r_i^{k,\varepsilon}, s_i^{k,\varepsilon}]$ for the corresponding interval. We will also use the notation $\tilde{\omega}_i^{k,\varepsilon}$ for $\omega_i^{k,\varepsilon}$ translated so that its starting point is ε and then truncated at level 0: with the notation of subsection 3.2.2, $\tilde{\omega}_i^{k,\varepsilon} = \mathrm{tr}_0 \circ \kappa_{(k+1)\varepsilon}(\omega_i^{k,\varepsilon}) \in \mathcal{S}_\varepsilon$.

Recall our notation Z_a for the total mass of the exit measure from $(-a, \infty)$. By the special Markov property, we know that, under $\mathbb{N}_0$ and conditionally on $Z_{k\varepsilon}$, or even on the σ-field $\mathcal{E}^{(-k\varepsilon,\infty)}$, $\mathcal{N}_k^\varepsilon$ is Poisson with intensity

$$Z_{k\varepsilon}\,\mathbb{N}_{-k\varepsilon}(\cdot).$$

Lemma 3.3.8. *Let $u \in D$, and let $0 < s_1 < s_2 < \sigma$ be determined by $p_\zeta(s_1) = p_\zeta(s_2) = u$. Then, for every sufficiently small ε in the sequence $(\varepsilon_n)_{n\geq 1}$, if $k_{u,\varepsilon} \geq 1$ is the integer determined by $-(k_{u,\varepsilon}+1)\varepsilon < V_u \leq -k_{u,\varepsilon}\varepsilon$, there exists a unique index $i_{u,\varepsilon} \in I^\varepsilon_{k_{u,\varepsilon}}$ such that*

$$(s_1, s_2) \subset (r^{k_{u,\varepsilon},\varepsilon}_{i_{u,\varepsilon}}, s^{k_{u,\varepsilon},\varepsilon}_{i_{u,\varepsilon}}),$$

and we have

$$\tilde{\omega}^{k_{u,\varepsilon},\varepsilon}_{i_{u,\varepsilon}} \longrightarrow \tilde{W}^{(u)}$$

as $\varepsilon \to 0$ along the sequence $(\varepsilon_n)_{n\geq 1}$.

Proof. Note that a priori we could have $k_{u,\varepsilon} = 0$, but this does not occur for ε small enough since $V_u < 0$. Then the index $i_{u,\varepsilon}$ is determined by the fact that the excursion $\omega^{k_{u,\varepsilon},\varepsilon}_{i_{u,\varepsilon}}$ corresponds to the descendants of the first ancestor of u at spatial position $-k_{u,\varepsilon}\varepsilon$. More specifically, the index $i_{u,\varepsilon}$ is determined by

$$r^{k_{u,\varepsilon},\varepsilon}_{i_{u,\varepsilon}} = \sup\{s \leq s_1 : \zeta_s \leq \tau_{-k_{u,\varepsilon}\varepsilon}(W_{s_1})\} \tag{3.17}$$

where we recall the notation $\tau_a(\mathrm{w}) = \inf\{t \geq 0 : \mathrm{w}(t) = a\}$. Since the image under p_ζ of the interval $(r^{k_{u,\varepsilon},\varepsilon}_{i_{u,\varepsilon}}, s^{k_{u,\varepsilon},\varepsilon}_{i_{u,\varepsilon}})$ corresponds to descendants of an ancestor of u, the inclusion

$$(s_1, s_2) \subset (r^{k_{u,\varepsilon},\varepsilon}_{i_{u,\varepsilon}}, s^{k_{u,\varepsilon},\varepsilon}_{i_{u,\varepsilon}})$$

is immediate. For the last property of the lemma, we first verify that

$$r^{k_{u,\varepsilon},\varepsilon}_{i_{u,\varepsilon}} \longrightarrow s_1\ , \quad s^{k_{u,\varepsilon},\varepsilon}_{i_{u,\varepsilon}} \longrightarrow s_2 \tag{3.18}$$

as $\varepsilon \to 0$ along the sequence $(\varepsilon_n)_{n\geq 1}$.

To this end, let s be such that $0 < s < s_1$, and observe that we have then

$$\inf_{r\in[s,s_1]} \zeta_r < \zeta_{s_1}$$

(otherwise u would be a branching point). On the other hand, for any $\gamma > 0$, there exists $\eta > 0$ such that $W_{s_1}(t) \geq V_u + \eta$ if $0 \leq t \leq \zeta_{s_1} - \gamma$ (by property (2) of the definition of an excursion debut, and the fact that a Brownian path cannot have two local minima at the same level). It follows that $\tau_{-k_{u,\varepsilon}\varepsilon}(W_{s_1}) \longrightarrow \zeta_{s_1}$ as $\varepsilon \to 0$, and together with (3.17) the preceding observations imply that $r^{k_{u,\varepsilon},\varepsilon}_{i_{u,\varepsilon}} > s$ for ε small enough, giving the desired convergence $r^{k_{u,\varepsilon},\varepsilon}_{i_{u,\varepsilon}} \longrightarrow s_1$. The proof of the other convergence $s^{k_{u,\varepsilon},\varepsilon}_{i_{u,\varepsilon}} \longrightarrow s_2$ is analogous.

Once we have obtained the convergences (3.18), we deduce the last assertion of the lemma from Lemma 3.2.8. With the notation of this lemma, we take $\omega' = W^{(u)}$ and $\omega^{(n)} = \kappa_{-V_u}(\omega^{k_n,\varepsilon_n}_{i_n})$, where we write $k_n = k_{u,\varepsilon_n}$ and $i_n = i_{u,\varepsilon_n}$ to simplify notation. We also take $\delta_n = -(k_n+1)\varepsilon_n - V_u \in (-\varepsilon_n, 0)$. The conclusion of the lemma then yields the fact that $\mathrm{tr}_{\delta_n}(\omega^{(n)})$ converges to $\mathrm{tr}_0(\omega') = \tilde{W}^{(u)}$. This is the result we need since one easily checks that $\mathrm{tr}_{\delta_n}(\omega^{(n)})$ coincides with $\tilde{\omega}^{k_n,\varepsilon_n}_{i_n}$ translated by δ_n. We still need to verify that assumptions (i)–(iii) of Lemma 3.2.8 hold with our choice of ω'. Assumptions (i) and (ii) hold by the definition of an excursion debut. Assumption (iii) holds because otherwise this would mean that there are two distinct local minimum times corresponding to the same local minimum of a path W_s, which is impossible. This completes the proof of the last assertion of the lemma. $\square$

Proof of Theorem 3.3.7. We consider two functions G and g as in the statement of the theorem, and we further assume that both G and g are bounded and continuous and take nonnegative values. Moreover we assume that g is nontrivial and is supported on a compact subinterval of $(-\infty, 0)$, and that there exists $\delta > 0$ such that $G(\omega) = 0$ if $\omega \notin \mathcal{S}^{(\delta)}$. By the latter assumption, the quantity $G(\tilde{W}^{(u)})$ is zero if $u \notin D_\delta$, and a fortiori if $u \notin D_{\delta/2}$. Since $D_{\delta/2}$ is finite (Lemma 3.3.6) we get, using the notation and the conclusion of Lemma 3.3.8,

$$\sum_{u \in D} g(V_u) G(\tilde{W}^{(u)}) = \sum_{u \in D_{\delta/2}} g(V_u) G(\tilde{W}^{(u)}) = \lim_{\varepsilon \to 0} \sum_{u \in D_{\delta/2}} g(-\varepsilon k_{u,\varepsilon}) G(\tilde{\omega}^{k_{u,\varepsilon},\varepsilon}_{i_{u,\varepsilon}}),$$

$\mathbb{N}_0$ a.e. (here and in the remaining part of the proof, we consider only values of ε in the sequence $(\varepsilon_n)_{n \geq 1}$, even if this is not mentioned explicitly). We next observe that we have

$$\sum_{u \in D_{\delta/2}} g(-\varepsilon k_{u,\varepsilon}) G(\tilde{\omega}^{k_{u,\varepsilon},\varepsilon}_{i_{u,\varepsilon}}) = \sum_{k=1}^{\infty} \sum_{i \in I_k^\varepsilon} g(-\varepsilon k)\, G(\tilde{\omega}_i^{k,\varepsilon}) \tag{3.19}$$

except on a set of $\mathbb{N}_0$-measure tending to 0 as $\varepsilon \to 0$. To see this, suppose that $\varepsilon < \delta/2$, and fix $k \geq 1$ and $i \in I_k^\varepsilon$. If $G(\tilde{\omega}_i^{k,\varepsilon}) \neq 0$, there exists a real $s \geq 0$ such that the path $W_s(\tilde{\omega}_i^{k,\varepsilon})$ hits level δ. This also means that there exists a real $s' \geq 0$ such that the path $W_{s'}(\omega_i^{k,\varepsilon})$ hits $-(k+1)\varepsilon + \delta$ before hitting $-(k+1)\varepsilon$, and we can take the smallest such real s'. Let s'' such that $W_{s''}(\omega_i^{k,\varepsilon})$ coincides with $W_{s'}(\omega_i^{k,\varepsilon})$ truncated at the (unique) time where it reaches its minimum before hitting $-(k+1)\varepsilon + \delta$ (in the tree coded by $\zeta(\omega_i^{k,\varepsilon})$, s'' corresponds to the unique ancestor with minimal spatial position of the vertex s'). Then it follows from our definitions that $u := p_\zeta(r_i^{k,\varepsilon} + s'')$ is an excursion debut, with $k_{u,\varepsilon} = k$ and $i_{u,\varepsilon} = i$ by construction, and the height of u is at least $\delta - \varepsilon > \delta/2$, so that $u \in D_{\delta/2}$. Thus any (nonzero) term appearing in the right-hand side of (3.19) also appears, at least once, in the left-hand side. To complete the proof of (3.19), we must still verify that no (nonzero) term in the right-hand side appears twice in the left-hand side. But this follows from the fact that the values of V_u for $u \in D$ are all distinct: since $D_{\delta/2}$ is finite, for ε small enough, there cannot be two distinct elements u, u' of $D_{\delta/2}$ such that V_u and $V_{u'}$ lie in the same interval $(-(k+1)\varepsilon, -k\varepsilon)$.

From the preceding considerations, we get that

$$\sum_{u \in D} g(V_u) G(\tilde{W}^{(u)}) = \lim_{\varepsilon \to 0} \sum_{k=1}^{\infty} \sum_{i \in I_k^\varepsilon} g(-\varepsilon k)\, G(\tilde{\omega}_i^{k,\varepsilon}),$$

in $\mathbb{N}_0$-measure. Here we notice that we can fix $\eta > 0$ such that $g(x) = 0$ if $x \geq -\eta$, and restrict our attention to the set $\{W_* \leq -\eta\}$, which has finite $\mathbb{N}_0$-measure. The next step is to deduce from the preceding convergence that we have also

$$\mathbb{N}_0\bigg(\sum_{u \in D} g(V_u) G(\tilde{W}^{(u)})\bigg) = \lim_{\varepsilon \to 0} \mathbb{N}_0\bigg(\sum_{k=1}^{\infty} \sum_{i \in I_k^\varepsilon} g(-\varepsilon k)\, G(\tilde{\omega}_i^{k,\varepsilon})\bigg). \tag{3.20}$$

For this, some uniform integrability is needed. For every integer $k \geq 1$, set

$$n_k^\varepsilon := \sum_{i \in I_k^\varepsilon} \mathbf{1}_{\{\|\tilde{\omega}_i^{k,\varepsilon}\| > \delta\}}.$$

Recalling our assumptions on g and G, we see that in order to deduce (3.20) from the preceding convergence, it suffices to verify that, for $p \in (1, 3/2)$, and for every $A > \eta$,

$$\mathbb{N}_0\left[\left(\sum_{k=\lfloor \eta/\varepsilon \rfloor+1}^{\lfloor A/\varepsilon \rfloor} n_k^\varepsilon\right)^p\right] \tag{3.21}$$

is bounded independently of ε. By the special Markov property, conditionally on the σ-field $\mathcal{E}^{(-k\varepsilon,\infty)}$, n_k^ε is Poisson with intensity $c_{\varepsilon,\delta}\, Z_{k\varepsilon}$, where $c_{\varepsilon,\delta} = \mathbb{N}_\varepsilon(\|\tilde{W}\| > \delta)$. In particular,

$$\mathbb{N}_0\Big((n_k^\varepsilon - c_{\varepsilon,\delta}\, Z_{k\varepsilon})^2 \,\Big|\, \mathcal{E}^{(-k\varepsilon,\infty)}\Big) = c_{\varepsilon,\delta}\, Z_{k\varepsilon}$$

and

$$\mathcal{M}_k^\varepsilon := \sum_{j=\lfloor \eta/\varepsilon \rfloor+1}^{k} (n_j^\varepsilon - c_{\varepsilon,\delta}\, Z_{j\varepsilon})\ , \quad k \geq \lfloor \eta/\varepsilon \rfloor$$

is a martingale with respect to the filtration $(\mathcal{E}^{(-(k+1)\varepsilon,\infty)})_{k \geq \lfloor \eta/\varepsilon \rfloor}$ – note that, by the construction of the truncated excursions $\tilde{\omega}_i^{k,\varepsilon}$, n_k^ε is $\mathcal{E}^{(-(k+1)\varepsilon,\infty)}$-measurable. The discrete Burkholder-Davis-Gundy inequalities (see e.g. [56, Théorème 5]) now give, for $p \in (1, 3/2)$ and for some constant C_p depending only on p,

$$\begin{aligned}
\mathbb{N}_0\Big(|\mathcal{M}^\varepsilon_{\lfloor A/\varepsilon \rfloor}|^p\Big) &\leq C_p\, \mathbb{N}_0\left[\left(\sum_{j=\lfloor \eta/\varepsilon \rfloor+1}^{\lfloor A/\varepsilon \rfloor} (\mathcal{M}_j^\varepsilon - \mathcal{M}_{j-1}^\varepsilon)^2\right)^{p/2}\right] \\
&\leq C_p \mathbb{N}_0(M_* < -\eta)^{1-p/2} \left(\mathbb{N}_0\left[\sum_{j=\lfloor \eta/\varepsilon \rfloor+1}^{\lfloor A/\varepsilon \rfloor} (\mathcal{M}_j^\varepsilon - \mathcal{M}_{j-1}^\varepsilon)^2\right]\right)^{p/2} \\
&= C_p \mathbb{N}_0(M_* < -\eta)^{1-p/2}\, c_{\varepsilon,\delta}^{p/2} \left(\mathbb{N}_0\left[\sum_{j=\lfloor \eta/\varepsilon \rfloor+1}^{\lfloor A/\varepsilon \rfloor} Z_{j\varepsilon}\right]\right)^{p/2} \\
&= C_p \mathbb{N}_0(M_* < -\eta)^{1-p/2}\, c_{\varepsilon,\delta}^{p/2}\, (\lfloor A/\varepsilon \rfloor - \lfloor \eta/\varepsilon \rfloor)^{p/2},
\end{aligned}$$

using Jensen's inequality (with respect to the probability measure $\mathbb{N}_0(\cdot \mid W_* < -\eta)$) in the second line, and in the last line the fact that $\mathbb{N}_0(Z_r) = 1$ for every $r > 0$ (see (3.10)). It follows from [45, Section 4] that there exists a constant $c_\delta < \infty$ such that $c_{\varepsilon,\delta} \leq c_\delta\, \varepsilon$ for every $\varepsilon < \delta/2$. Hence we get that the quantities $\mathbb{N}_0(|\mathcal{M}^\varepsilon_{\lfloor A/\varepsilon \rfloor}|^p)$ are uniformly bounded when $\varepsilon < \delta/2$. Finally, we write

$$\sum_{k=\lfloor \eta/\varepsilon \rfloor+1}^{\lfloor A/\varepsilon \rfloor} n_k^\varepsilon = \mathcal{M}^\varepsilon_{\lfloor A/\varepsilon \rfloor} + c_{\varepsilon,\delta} \sum_{k=\lfloor \eta/\varepsilon \rfloor+1}^{\lfloor A/\varepsilon \rfloor} Z_{\varepsilon k}$$

and we use again the bound $c_{\varepsilon,\delta} \leq c_\delta\, \varepsilon$ together with the fact that the random variables Z_a, $0 < a \leq A$ are bounded in $L^p(\mathbb{N}_0)$ when $1 < p < 3/2$ (Lemma 3.2.11). This gives us the desired bound for the quantities in (3.21), and justifies the passage to the limit under the integral in (3.20) – incidentally this also shows that the left-hand side of (3.20) is a finite quantity.

We then use the special Markov property once again to obtain

$$\mathbb{N}_0\left(\sum_{k=1}^{\infty} \sum_{i \in I_k^\varepsilon} g(-\varepsilon k)\, G(\tilde{\omega}_i^{k,\varepsilon})\right) = \sum_{k=1}^{\infty} g(-\varepsilon k) \mathbb{N}_0\big(Z_{-k\varepsilon}\, \mathbb{N}_\varepsilon(G(\tilde{W}))\big) = \left(\sum_{k=1}^{\infty} g(-\varepsilon k)\right) \mathbb{N}_\varepsilon(G(\tilde{W})).$$

Now note that

$$\varepsilon \sum_{k=1}^{\infty} g(-\varepsilon k) \underset{\varepsilon\to 0}{\longrightarrow} \int_{-\infty}^{0} g(x)\,\mathrm{d}x,$$

and so we deduce from (3.20) and the preceding two displays that

$$\varepsilon^{-1}\mathbb{N}_\varepsilon(G(\tilde{W})) \underset{\varepsilon\to 0}{\longrightarrow} K_G$$

where the limit $K_G < \infty$ is such that

$$\mathbb{N}_0\left(\sum_{u\in D} g(V_u) G(\tilde{W}^{(u)})\right) = K_G \int_{-\infty}^{0} g(x)\,\mathrm{d}x.$$

We now set, for every measurable subset F of $\mathcal{S}$,

$$\mathbb{N}_0^*(F) := \frac{\mathbb{N}_0\Big(\sum_{u\in D} g(V_u)\mathbf{1}_F(\tilde{W}^{(u)})\Big)}{\int_{-\infty}^{0} g(x)\,\mathrm{d}x}.$$

This defines a positive measure on $\mathcal{S}$, which is supported on $\mathcal{S}_0$ since $\tilde{W}^{(u)} \in \mathcal{S}_0$ for every $u \in D$. Furthermore, if G satisfies the assumptions listed at the beginning of the proof, we have

$$\mathbb{N}_0^*(G) = K_G < \infty,$$

and this implies that the sets $\mathcal{S}^{(\delta)}$ have a finite $\mathbb{N}_0^*$-measure. Since it is clear that $\mathbb{N}_0^*(\|\omega\| = 0) = 0$, we get that $\mathbb{N}_0^*$ is σ-finite. Finally, still under the previous assumptions on G, we have also obtained that

$$\mathbb{N}_0^*(G) = \lim_{\varepsilon\to 0} \varepsilon^{-1}\mathbb{N}_\varepsilon(G(\tilde{W})),$$

which proves (3.16) (note that we considered a fixed sequence of values of ε, but the same would hold for any such sequence). This completes the proof. $\square$

Recall the notation $M(\omega) = \sup\{\omega_s(t) : s \geq 0, 0 \leq s \leq \zeta_s\}$. We also set

$$\tilde{M} = M(\tilde{W}) = \sup\{W_s(t) : s \geq 0, t \leq \zeta_s \wedge \tau_0^*(W_s)\}.$$

We can derive the distribution of M under $\mathbb{N}_0^*$.

Lemma 3.3.9. *For every $\delta > 0$, we have*

$$\mathbb{N}_0^*(M > \delta) = c_0\, \delta^{-3},$$

where the constant c_0 is given by

$$c_0 = 3\,\pi^{-3/2}\,\Gamma(\frac{1}{3})^3\Gamma(\frac{7}{6})^3.$$

Proof. It follows from [45, Section 4] (see in particular formula (9) in [45]) that

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}\mathbb{N}_\varepsilon(\tilde M > \delta) = c_0\, \delta^{-3} \tag{3.22}$$

where the constant c_0 is as in the lemma. On the other hand, we know that

$$\mathbb{N}^*_0(G) = \lim_{\varepsilon\to 0} \varepsilon^{-1}\mathbb{N}_\varepsilon(G(\tilde W)),$$

for any bounded continuous function G vanishing on the complement of $\mathcal{S}^{(\eta)}$ for some $\eta > 0$. Noting that the limit in (3.22) depends continuously on δ, we can approximate the indicator function of the set $\{M > \delta\}$ by such functions G, and obtain

$$\mathbb{N}^*_0(M > \delta) = \lim_{\varepsilon\to 0} \varepsilon^{-1}\mathbb{N}_\varepsilon(\tilde M > \delta) = c_0\, \delta^{-3}.$$

This completes the proof. □

We may now restate the last assertion of Theorem 3.3.7 in a way more suitable for our applications.

Corollary 3.3.10. *Let $\delta > 0$. As $\varepsilon \to 0$, the law of $\tilde W$ under $\mathbb{N}_\varepsilon(\cdot \mid \tilde M > \delta)$ converges weakly to $\mathbb{N}^*_0(\cdot \mid M > \delta)$.*

Proof. Let G be bounded and continuous on $\mathcal{S}$ and such that $G(\omega) = 0$ if $\omega \notin \mathcal{S}^{(\delta)}$. Then, for $\varepsilon \in (0, \delta)$,

$$\mathbb{N}_\varepsilon\Big(G(\tilde W)\,\Big|\, \tilde M > \delta\Big) = \frac{\mathbb{N}_\varepsilon(G(\tilde W))}{\mathbb{N}_\varepsilon(\tilde M > \delta)} \underset{\varepsilon\to 0}{\longrightarrow} \frac{\mathbb{N}^*_0(G)}{\mathbb{N}^*_0(M > \delta)} = \mathbb{N}^*_0\Big(G\,\Big|\, M > \delta\Big),$$

using (3.16) and (3.22). The desired result follows. □

We conclude this section by deriving a useful scaling property of $\mathbb{N}^*_0$. For $\lambda > 0$, for every $\omega \in \mathcal{S}$, we define $\theta_\lambda(\omega) \in \mathcal{S}$ by $\theta_\lambda(\omega) = \omega'$, with

$$\omega'_s(t) = \sqrt{\lambda}\, \omega_{s/\lambda^2}(t/\lambda)\,, \qquad \text{for } s \geq 0,\ 0 \leq t \leq \zeta'_s = \lambda \zeta_{s/\lambda^2}.$$

Note that, for every $x \geq 0$, $\theta_\lambda(\mathbb{N}_x) = \lambda\, \mathbb{N}_{x\sqrt{\lambda}}$. The measure $\mathbb{N}^*_0$ enjoys a similar scaling property.

Lemma 3.3.11. *For every $\lambda > 0$, $\theta_\lambda(\mathbb{N}^*_0) = \lambda^{3/2}\, \mathbb{N}^*_0$.*

Proof. Let G be a function on $\mathcal{S}$ satisfying the conditions required for (3.16). Then,

$$\begin{aligned}
\mathbb{N}^*_0(G) &= \lim_{\varepsilon\to 0} \varepsilon^{-1}\mathbb{N}_\varepsilon(G(\tilde W)) \\
&= \lim_{\varepsilon\to 0} \varepsilon^{-1}\lambda^{-1}\mathbb{N}_{\varepsilon/\sqrt{\lambda}}(G(\theta_\lambda(\tilde W))) \\
&= \lim_{\varepsilon\to 0} \lambda^{-3/2} \times (\varepsilon/\sqrt{\lambda})^{-1}\mathbb{N}_{\varepsilon/\sqrt{\lambda}}(G(\theta_\lambda(\tilde W))) \\
&= \lambda^{-3/2}\, \mathbb{N}^*_0(G \circ \theta_\lambda)
\end{aligned}$$

giving the desired result. □

3.4 An almost sure construction

In this section, we fix $\delta > 0$ and we give an almost sure construction of a snake trajectory distributed according to $\mathbb{N}_0^*(\cdot \mid M > \delta)$. This construction will be useful later when we discuss exit measures.

Let $0 < \varepsilon < \varepsilon' < \delta$, and let $W^{\delta,\varepsilon}$ be a random snake trajectory distributed according to $\mathbb{N}_\varepsilon(\cdot \mid \tilde{M} > \delta)$. Consider the excursions of $W^{\delta,\varepsilon}$ outside the interval $(0,\varepsilon')$. The conditioning on $\{\tilde{M} > \delta\}$ implies that there is at least one such excursion ω' starting from ε' and such that $\tilde{M}(\omega') > \delta$. Furthermore, if we pick uniformly at random one of the excursions ω' starting from ε' that satisfy $\tilde{M}(\omega') > \delta$, the special Markov property ensures that this excursion will be distributed according to $\mathbb{N}_{\varepsilon'}(\cdot \mid \tilde{M} > \delta)$. For $\omega \in \mathcal{S}_\varepsilon$ such that $\tilde{M}(\omega) > \delta$, let $\Theta_{\varepsilon,\varepsilon'}(\omega, \mathrm{d}\omega')$ be the probability measure on $\mathcal{S}_{\varepsilon'}$ defined as the law of an excursion of ω outside $(0,\varepsilon')$ chosen uniformly at random among those excursions that satisfy $\tilde{M} > \delta$. Then, the preceding considerations show that the second marginal of the probability measure $\Pi_{\varepsilon,\varepsilon'}$ defined on $\mathcal{S}_\varepsilon \times S_{\varepsilon'}$ by

$$\Pi_{\varepsilon,\varepsilon'}(\mathrm{d}\omega\,\mathrm{d}\omega') = \mathbb{N}_\varepsilon(\mathrm{d}\omega \mid \tilde{M} > \delta)\,\Theta_{\varepsilon,\varepsilon'}(\omega, \mathrm{d}\omega')$$

is $\mathbb{N}_{\varepsilon'}(\cdot \mid \tilde{M} > \delta)$.

Now let $(\varepsilon_n)_{n\geq 1}$ be a sequence of positive reals in $(0,\delta)$ decreasing to 0. We claim that we can construct, on a suitable probability space, a sequence $(W^{\delta,\varepsilon_n})_{n\geq 1}$ of random variables with values in $\mathcal{S}$ such that the following holds:

(i) For every $n \geq 1$, W^{δ,ε_n} is distributed according to $\mathbb{N}_{\varepsilon_n}(\cdot \mid \tilde{M} > \delta)$.

(ii) For every $1 \leq n < m$, W^{δ,ε_n} is an excursion of W^{δ,ε_m} outside $(0,\varepsilon_n)$.

Indeed, we use the Kolmogorov extension theorem to construct the sequence $(W^{\delta,\varepsilon_n})_{n\geq 1}$ so that, for every $n \geq 1$, the law of $(W^{\delta,n}, W^{\delta,n-1}, \ldots, W^{\delta,1})$ is

$$\mathbb{N}_{\varepsilon_n}(\mathrm{d}\omega_n \mid \tilde{M} > \delta)\,\Theta_{\varepsilon_n,\varepsilon_{n-1}}(\omega_n, \mathrm{d}\omega_{n-1})\Theta_{\varepsilon_{n-1},\varepsilon_{n-2}}(\omega_n, \mathrm{d}\omega_{n-2})\ldots\Theta_{\varepsilon_2,\varepsilon_1}(\omega_2, \mathrm{d}\omega_1)$$

and properties (i) and (ii) hold by construction.

We set $\tilde{W}^{\delta,\varepsilon_n} = \mathrm{tr}_0(W^{\delta,\varepsilon_n})$ for every n. Clearly, it is still true that, for $1 \leq n < m$, $\tilde{W}^{\delta,\varepsilon_n}$ is an excursion of $\tilde{W}^{\delta,\varepsilon_m}$ outside $(0,\varepsilon_n)$. Therefore, for every $1 \leq n < m$, $\tilde{W}^{\delta,\varepsilon_n}$ is a subtrajectory of $\tilde{W}^{\delta,\varepsilon_m}$ and we write $[a_{n,m}, b_{n,m}] \subset [0,\sigma_m]$ for the associated interval. Note that $b_{n,m} - a_{n,m} = \sigma_n$, where $\sigma_n = \sigma(\tilde{W}^{\delta,\varepsilon_n})$ is the duration of $\tilde{W}^{\delta,\varepsilon_n}$. Furthermore, if $1 \leq n < m < \ell$, we have $[a_{n,\ell}, b_{n,\ell}] \subset [a_{m,\ell}, b_{m,\ell}]$, and more precisely

$$a_{n,\ell} = a_{n,m} + a_{m,\ell}\,, \tag{3.23}$$

$$\sigma_\ell - b_{n,\ell} = (\sigma_m - b_{n,m}) + (\sigma_\ell - b_{m,\ell})\,. \tag{3.24}$$

In particular, for n fixed, the sequence $(a_{n,m})_{m>n}$ is increasing, and we denote its limit by $a_{n,\infty}$ (the fact that this limit is finite will be obtained at the beginning of the proof of the next proposition).

Proposition 3.4.1. *We have a.s.*

$$\tilde{W}^{\delta,\varepsilon_n} \underset{n\to\infty}{\longrightarrow} W^{\delta,0}, \quad \text{in } \mathcal{S}\,,$$

where the a.s. limit $W^{\delta,0}$ is distributed according to $\mathbb{N}_0^*(\cdot \mid M > \delta)$. Furthermore, $\tilde{W}^{\delta,\varepsilon_n}$ is a subtrajectory of $W^{\delta,0}$, for every $n \geq 1$, and $\sigma(\tilde{W}^{\delta,\varepsilon_n}) \uparrow \sigma(W^{\delta,0})$ as $n \to \infty$.

Proof. By Corollary 3.3.10, we already know that the sequence $(\tilde{W}^{\delta,\varepsilon_n})_{n\geq 1}$ converges in distribution to $\mathbb{N}_0^*(\cdot \mid M > \delta)$, and in particular $\sigma_n = \sigma(\tilde{W}^{\delta,\varepsilon_n})$ converges in distribution to the law of σ under $\mathbb{N}_0^*(\cdot \mid M > \delta)$. On the other hand, the sequence $(\sigma_n)_{n\geq 1}$ is increasing and thus has an a.s. limit σ_∞. We conclude that σ_∞ is distributed as σ under $\mathbb{N}_0^*(\cdot \mid M > \delta)$, and in particular, $\sigma_\infty < \infty$ a.s.

Since $a_{n,m} \leq \sigma_m - \sigma_n$ if $n < m$, we obtain that, for every n,

$$a_{n,\infty} \leq \sigma_\infty - \sigma_n.$$

It follows that

$$\lim_{n\to\infty} a_{n,\infty} = 0, \text{ a.s.} \tag{3.25}$$

Then, for every fixed n, $b_{n,m} = a_{n,m} + \sigma_n$ converges as $m \uparrow \infty$ to $b_{n,\infty} = a_{n,\infty} + \sigma_n$, and, from (3.23) and (3.24), we have, for $n < m$,

$$a_{n,\infty} = a_{n,m} + a_{m,\infty}\ , \qquad \sigma_\infty - b_{n,\infty} = (\sigma_\infty - b_{m,\infty}) + (\sigma_m - b_{n,m})\,. \tag{3.26}$$

Set $\tilde{\zeta}_s^{\delta,\varepsilon_n} = \zeta_s(\tilde{W}^{\delta,\varepsilon_n})$ to simplify notation. By the definition of subtrajectories we know that $\tilde{\zeta}_s^{\delta,\varepsilon_n} = \tilde{\zeta}^{\delta,\varepsilon_m}_{(a_{n,m}+s)\wedge b_{n,m}} - \tilde{\zeta}^{\delta,\varepsilon_m}_{a_{n,m}}$ if $n < m$. We claim that we have a.s.

$$\lim_{n\to\infty} \left(\sup_{m>n} \Big(\sup_{0\leq s\leq a_{n,m}} \tilde{\zeta}_s^{\delta,\varepsilon_m} \Big) \right) = 0 \tag{3.27}$$

To verify this claim, first observe that, if $n < n' < m$, we have

$$\sup_{0\leq s\leq a_{n',m}} \tilde{\zeta}_s^{\delta,\varepsilon_m} \leq \sup_{0\leq s\leq a_{n,m}} \tilde{\zeta}_s^{\delta,\varepsilon_m}$$

because $a_{n',m} \leq a_{n,m}$. It then follows that the supremum over $m > n$ in (3.27) is a decreasing function of n, and so the limit in the left-hand side of (3.27) exists a.s. as a decreasing limit. Call L this limit. We argue by contradiction assuming that $P(L > 0) > 0$. Then we choose $\xi > 0$ such that $P(L > \xi) > 0$, and we note that, on the event $\{L > \xi\}$, we can find a sequence $n_1 < m_1 < n_2 < m_2 < \cdots$, such that, for every $i = 1, 2, \ldots$, we have

$$\sup_{0\leq s\leq a_{n_i,m_i}} \tilde{\zeta}_s^{\delta,\varepsilon_{m_i}} > \xi.$$

It then follows that, on the same event $\{L > \xi\}$ of positive probability, for any integer $k \geq 1$, and for every large enough n, there exist k disjoint intervals $[r_1, s_1], \ldots, [r_k, s_k]$ such that $\tilde{\zeta}_{s_i}^{\delta,\varepsilon_n} - \tilde{\zeta}_{r_i}^{\delta,\varepsilon_n} > \xi$ for every $1 \leq i \leq k$. The latter property contradicts the tightness of the sequence of the laws of $\tilde{W}^{\delta,\varepsilon_n}$ in $\mathcal{S}$, and this contradiction proves our claim (3.27).

By the same argument, we have also

$$\lim_{n\to\infty} \left(\sup_{m>n} \Big(\sup_{b_{n,m}\leq s\leq \sigma_m} \tilde{\zeta}_s^{\delta,\varepsilon_m} \Big) \right) = 0. \tag{3.28}$$

We can now use (3.27) and (3.28) to verify that $(\tilde{\zeta}_s^{\delta,\varepsilon_n})_{s\geq 0}$ converges uniformly as $n \to \infty$, a.s. To this end, we define

$$\zeta_s^{(n)} = \begin{cases} 0 & \text{if } s \leq a_{n,\infty}, \\ \tilde{\zeta}^{\delta,\varepsilon_n}_{s-a_{n,\infty}} & \text{if } a_{n,\infty} \leq s \leq b_{n,\infty}, \\ 0 & \text{if } s \geq b_{n,\infty}. \end{cases}$$

Recalling the formula $\tilde{\zeta}^{\delta,\varepsilon_n}_s = \tilde{\zeta}^{\delta,\varepsilon_m}_{(a_{n,m}+s)\wedge b_{n,m}} - \tilde{\zeta}^{\delta,\varepsilon_m}_{a_{n,m}}$, and using (3.26), we get for $n < m$,

$$\sup_{s\geq 0} |\zeta^{(n)}_s - \zeta^{(m)}_s| \leq \sup_{0\leq s\leq a_{n,m}} \tilde{\zeta}^{\delta,\varepsilon_m}_s + \sup_{b_{n,m}\leq s\leq \sigma_m} \tilde{\zeta}^{\delta,\varepsilon_m}_s,$$

and the right-hand side tends to 0 a.s. as n and m tend to ∞ with $n < m$, by (3.27) and (3.28). This gives the a.s. uniform convergence of $(\zeta^{(n)}_s)_{s\geq 0}$ as $n \to \infty$. Write $(\zeta^{\delta,0}_s)_{s\geq 0}$ for the limit. The a.s. uniform convergence of $(\tilde{\zeta}^{\delta,\varepsilon_n}_s)_{s\geq 0}$ toward the same limit $(\zeta^{\delta,0}_s)_{s\geq 0}$ then follows using now (3.25), and we have also $\sup\{s \geq 0 : \tilde{\zeta}^{\delta,\varepsilon_n}_s > 0\} = \sigma_n \longrightarrow \sigma_\infty = \sup\{s \geq 0 : \zeta^{\delta,0}_s > 0\}$ as $n \to \infty$.

Let $\Gamma^{\delta,\varepsilon_n}_s$ stand for the endpoint of the path $\widetilde{W}^{\delta,\varepsilon_n}_s$. Very similar arguments show that the analogs of (3.27) and (3.28) where $\tilde{\zeta}^{\delta,\varepsilon_m}_s$ is replaced by $\Gamma^{\delta,\varepsilon_m}_s$ hold, and it follows that $(\Gamma^{\delta,\varepsilon_n}_s)_{s\geq 0}$ also converges uniformly to a limit denoted by $(\Gamma^{\delta,0}_s)_{s\geq 0}$, a.s. The pair $(\zeta^{\delta,0}, \Gamma^{\delta,0})$ is then a random tree-like path, and letting $W^{\delta,0}$ be the associated snake trajectory, we have obtained that $\tilde{W}^{\delta,\varepsilon_n}$ converges a.s. to $W^{\delta,0}$. Since we know that $\tilde{W}^{\delta,\varepsilon_n}$ converges in distribution to $\mathbb{N}^*_0(\cdot \mid M > \delta)$, $W^{\delta,0}$ is distributed according to $\mathbb{N}^*_0(\cdot \mid M > \delta)$.

Finally, it follows from our construction that, for every $n \geq 1$, $\tilde{W}^{\delta,\varepsilon_n}$ is the subtrajectory of $W^{\delta,0}$ associated with the interval $[a_{n,\infty}, b_{n,\infty}]$, and the property $\sigma(\tilde{W}^{\delta,\varepsilon_n}) \uparrow \sigma(W^{\delta,0})$ is just the fact that $\sigma_n \uparrow \sigma_\infty$. This completes the proof. □

3.5 The re-rooting representation

In this section, we provide a formula connecting the measures $\mathbb{N}_0$ and $\mathbb{N}^*_0$ via a re-rooting technique. We first need to introduce some notation.

Recall the re-rooting operator R_s from subsection 3.2.2. For every $\omega \in \mathcal{S}_0$, for every $s \in [0, \sigma(\omega)]$, we set

$$W^{[s]}(\omega) = \kappa_{-\hat{W}_s(\omega)} \circ R_s(\omega).$$

In other words, $W^{[s]}(\omega)$ is just ω re-rooted at s and then shifted so that the spatial position of the root is again 0. Note that we slightly abuse notation here because it would have been more consistent with the notation of subsection 3.2.2 to take $W^{[s]}(\omega) = R_s(\omega)$.

Theorem 3.5.1. *For every nonnegative measurable function G on $\mathcal{S}$, the following equality holds.*

$$\mathbb{N}^*_0\Big(\int_0^\sigma \mathrm{d}r\, G(W^{[r]})\Big) = 2\,\mathbb{N}_0\Big(\int_0^\infty \mathrm{d}b\, G(\mathrm{tr}_{-b}(W))Z_b\Big)$$

where we recall that Z_b stands for the total mass of the exit measure outside $(-b, \infty)$.

Proof. We start from the re-rooting theorem in [51, Theorem 2.3]. For every nonnegative measurable function F on $\mathbb{R}_+ \times \mathcal{S}$,

$$\mathbb{N}_0\Big(\int_0^\sigma \mathrm{d}s\, F(s, W^{[s]})\Big) = \mathbb{N}_0\Big(\int_0^\sigma \mathrm{d}s\, F(s, W)\Big) \tag{3.29}$$

We apply this result to a function F of the form

$$F(s,\omega) = G\big(\mathrm{tr}_{\underline{\omega}_{\sigma-s}}(\omega)\big)\, g(\underline{\omega}_{\sigma-s} - \hat{\omega}_{\sigma-s}),$$

where we recall the notation $\underline{\mathrm{w}} = \min\{\mathrm{w}(t) : 0 \le t \le \zeta_{(\mathrm{w})}\}$, and we suppose that G and g satisfy the assumptions stated at the beginning of the proof of Theorem 3.3.7, and the additional assumption that there exists a constant $K > 0$ such that $G(\omega) = 0$ if $\|\omega\| \ge K$. We note that our definitions give under $\mathbb{N}_0$,

$$\begin{aligned}\hat{W}^{[s]}_{\sigma-s} &= -\hat{W}_s\\ \underline{W}^{[s]}_{\sigma-s} &= \underline{W}_s - \hat{W}_s.\end{aligned}$$

Consequently, we have

$$F(s, W^{[s]}) = G\big(\mathrm{tr}_{\underline{W}_s - \hat{W}_s}(W^{[s]})\big)\, g(\underline{W}_s).$$

We can then decompose the integral

$$\int_0^\sigma \mathrm{d}s\, F(s, W^{[s]})$$

as a sum over the sets $\{s \in [0,\sigma] : p_\zeta(s) \in C_u\}$ where u varies over D. These sets cover $[0,\sigma]$ (except for a Lebesgue negligible subset) and they are pairwise disjoint. Furthermore, if $u \in D$, it follows from our definitions that we have $\underline{W}_s = V_u$ for every $s \in [0,\sigma]$ such that $p_\zeta(s) \in C_u$, and

$$\int_{\{s\in[0,\sigma]:p_\zeta(s)\in C_u\}} \mathrm{d}s\, G\big(\mathrm{tr}_{\underline{W}_s - \hat{W}_s}(W^{[s]})\big) = H(\tilde{W}^{(u)}),$$

where

$$H(\omega) = \int_0^\sigma \mathrm{d}r\, G(W^{[r]}(\omega)).$$

Summarizing, the left-hand side of (3.29) is equal to

$$\mathbb{N}_0\bigg(\sum_{u\in D} g(V_u)\, H(\tilde{W}^{(u)})\bigg) = \Big(\int_{-\infty}^0 g(x)\,\mathrm{d}x\Big)\, \mathbb{N}_0^*(H) \tag{3.30}$$

by Theorem 3.3.7.

On the other hand, the right-hand side of (3.29) is equal to

$$\mathbb{N}_0\Big(\int_0^\sigma \mathrm{d}s\, G(\mathrm{tr}_{\underline{W}_s}(W))\, g(\underline{W}_s - \hat{W}_s)\Big).$$

We can evaluate this quantity via a discrete approximation. Using Lemma 3.2.7, we have $\mathbb{N}_0$ a.e.

$$\int_0^\sigma \mathrm{d}s\, G(\mathrm{tr}_{\underline{W}_s}(W))\, g(\underline{W}_s - \hat{W}_s) = \lim_{n\to\infty} \int_0^\sigma \mathrm{d}s\, g(\underline{W}_s - \hat{W}_s) \sum_{k=1}^\infty \mathbf{1}_{\{\underline{W}_s \in (-(k+1)/n, -k/n]\}}\, G(\mathrm{tr}_{-k/n}(W)),$$

and we note that, if g is supported on $[-A, 0]$, the quantities in the right-hand side are bounded independently of $n \ge 1$ by a constant times

$$\int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\underline{W}_s \ge -K-1\}} \mathbf{1}_{\{\underline{W}_s - \hat{W}_s \ge -A\}}.$$

The point is that if $s \in [0,\sigma]$ is such that $\underline{W}_s < -K-1$, then the unique integer k such that $\underline{W}_s \in (-(k+1)/n, -k/n]$ also satisfies $-k/n < -K$ and we have $G(\mathrm{tr}_{-k/n}(W)) = 0$ by our assumption on G. The quantity in the last display is integrable under $\mathbb{N}_0$ as a simple application of the first-moment formula for the Brownian snake (3.6). This makes it possible to use dominated convergence and to get that

$$\begin{aligned}&\mathbb{N}_0\Big(\int_0^\sigma \mathrm{d}s\, G(\mathrm{tr}_{\underline{W}_s}(W))\, g(\underline{W}_s - \hat{W}_s)\Big) \\ &= \lim_{n\to\infty}\sum_{k=1}^{\infty} \mathbb{N}_0\Big(\int_0^\sigma \mathrm{d}s\, g(\underline{W}_s - \hat{W}_s)\, \mathbf{1}_{\{\underline{W}_s\in(-(k+1)/n,-k/n]\}}\, G(\mathrm{tr}_{-k/n}(W))\Big).\end{aligned} \tag{3.31}$$

Then, for every integer $k \geq 1$, an application of the special Markov property (note that $G(\mathrm{tr}_{-k/n}(W))$ is $\mathcal{E}^{(-k/n,\infty)}$-measurable by the very definition of this σ-field) gives

$$\begin{aligned}&\mathbb{N}_0\Big(\int_0^\sigma \mathrm{d}s\, g(\underline{W}_s - \hat{W}_s)\, \mathbf{1}_{\{\underline{W}_s\in(-(k+1)/n,-k/n]\}}\, G(\mathrm{tr}_{-k/n}(W))\Big) \\ &= \mathbb{N}_0\bigg(Z_{k/n}\, G(\mathrm{tr}_{-k/n}(W))\, \mathbb{N}_{-k/n}\Big(\int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\underline{W}_s>-(k+1)/n\}}\, g(\underline{W}_s - \hat{W}_s)\Big)\bigg) \\ &= \mathbb{N}_0\big(Z_{k/n}\, G(\mathrm{tr}_{-k/n}(W))\big) \times \mathbb{N}_{-k/n}\Big(\int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\underline{W}_s>-(k+1)/n\}}\, g(\underline{W}_s - \hat{W}_s)\Big) \\ &= \mathbb{N}_0\big(Z_{k/n}\, G(\mathrm{tr}_{-k/n}(W))\big)\, \mathbb{E}_{-k/n}\Big[\int_0^\infty \mathrm{d}t\, \mathbf{1}_{\{\min\{B_r:0\le r\le t\}>-(k+1)/n\}}\, g(\min\{B_r : 0 \le r \le t\} - B_t)\Big] \\ &= \frac{2}{n}\Big(\int_{-\infty}^0 \mathrm{d}x\, g(x)\Big)\, \mathbb{N}_0\big(Z_{k/n}\, G(\mathrm{tr}_{-k/n}(W))\big),\end{aligned}$$

using again the first-moment formula for the Brownian snake (3.6) in the third equality and in the last equality the fact that, if $b < a$,

$$\mathbb{E}_a\bigg(\int_0^{\inf\{t:B_t=b\}} \mathrm{d}t f\,(B_t - \min\{B_r : 0 \le r \le t\})\bigg) = 2(a-b)\int_0^\infty \mathrm{d}x f(x).$$

The latter formula can be derived from Lévy's theorem stating that $B_t - \min\{B_r : 0 \le r \le t\}$ is a reflected Brownian motion, together with standard excursion theory and the formula

$$\int n(dh)\int_0^\sigma \mathrm{d}s f(h(s)) = 2\int_0^\infty \mathrm{d}x f(x).$$

From (3.31), we then deduce that

$$\begin{aligned}\mathbb{N}_0\Big(\int_0^\sigma \mathrm{d}s\, G(\mathrm{tr}_{\underline{W}_s}(W))\, g(\underline{W}_s - \hat{W}_s)\Big) &= \lim_{n\to\infty}\frac{2}{n}\Big(\int_{-\infty}^0 \mathrm{d}x\, g(x)\Big)\sum_{k=1}^{\infty}\mathbb{N}_0\big(Z_{k/n}\, G(\mathrm{tr}_{-k/n}(W))\big) \\ &= 2\Big(\int_{-\infty}^0 \mathrm{d}x\, g(x)\Big)\, \mathbb{N}_0\Big(\int_0^\infty \mathrm{d}b\, Z_b\, G(\mathrm{tr}_{-b}(W))\Big),\end{aligned}$$

where the last equality is justified by Lemma 3.2.7 together with our assumptions on G and the integrablity properties of the exit measure process Z that were already used in the proof of Theorem 3.3.7. Finally, the equality between the right-hand side of the last display and the right-hand side of (3.30) gives the statement of the theorem.

□

3.6 The exit measure

We now define the exit measure from $(0, \infty)$ under $\mathbb{N}_0^*$. Informally, this exit measure corresponds to the quantity of snake trajectories that return to 0.

Proposition 3.6.1. *The limit*

$$\lim_{\varepsilon \to 0} \varepsilon^{-2} \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\hat{W}_s < \varepsilon\}}$$

exists in probability under $\mathbb{N}_0^*(\cdot \mid \sigma > \eta)$, for every $\eta > 0$, and defines a finite random variable denoted by Z_0^*.

Proof. We rely on the re-rooting property of the preceding section. Let $(\varepsilon_n)_{n \geq 1}$ be a sequence of positive reals converging to 0. Recalling Lemma 3.2.10 and the subsequent remarks, we can extract from the sequence $(\varepsilon_n)_{n \geq 1}$ a subsequence $(\beta_n)_{n \geq 1}$ such that, for every $b < 0$,

$$Z_b = \lim_{n \to \infty} \beta_n^{-2} \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\zeta_s \leq \tau_{-b}(W_s), \hat{W}_s < -b + \beta_n\}}\,, \qquad \mathbb{N}_0 \text{ a.e.} \tag{3.32}$$

Then, for $\omega \in \mathcal{S}$, we set $G(\omega) = 0$ if the limit

$$\lim_{n \to \infty} \beta_n^{-2} \int_0^{\sigma(\omega)} \mathrm{d}s\, \mathbf{1}_{\{\hat{W}_s(\omega) < W_*(\omega) + \beta_n\}}$$

exists (and is finite), and $G(\omega) = 1$ otherwise. By (3.32), we have $G(\mathrm{tr}_{-b}(W)) = 0$, $\mathbb{N}_0$ a.e. on the event $\{W_* \leq -b\} = \{Z_b > 0\}$, for every $b > 0$. By Theorem 3.5.1, we have then

$$\mathbb{N}_0^*\left(\int_0^\sigma \mathrm{d}s\, G(W^{[s]}) \right) = 0.$$

We have thus obtained that $\mathbb{N}_0^*$ a.e., for Lebesgue a.e. $r \in [0, \sigma]$, $G(W^{[r]}) = 0$. By considering just one value of r for which $G(W^{[r]}) = 0$, this says that the convergence of the proposition holds $\mathbb{N}_0^*$ a.e. along the sequence $(\beta_n)_{n \geq 1}$. We have thus shown that from any sequence of positive real numbers converging to 0 we can extract a subsequence along which the convergence of the proposition holds $\mathbb{N}_0^*$ a.e. The statement of the proposition follows. $\square$

Recall from subsection 3.2.5 that we have fixed a sequence $(\alpha_n)_{n \geq 1}$ such that (3.32) holds. This allows us to define $Z_0^*(\omega)$ for *every* $\omega \in \mathcal{S}$, by setting

$$Z_0^*(\omega) = \liminf_{n \to \infty} \alpha_n^{-2} \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\hat{W}_s(\omega) < W_*(\omega) + \alpha_n\}}, \tag{3.33}$$

noting that the liminf is a limit $\mathbb{N}_0^*$ a.e. (and of course we can replace $W_*(\omega)$ by 0, $\mathbb{N}_0^*$ a.e.).

Our next goal is to compute the joint distribution of the pair (Z_0^*, σ) under $\mathbb{N}_0^*$.

Proposition 3.6.2. *The distribution of the pair (Z_0^*, σ) under $\mathbb{N}_0^*$ has a density f given for $z > 0$ and $s > 0$ by*

$$f(z, s) = \frac{\sqrt{3}}{2\pi} \sqrt{z}\, s^{-5/2} \exp\left(-\frac{z^2}{2s} \right).$$

In particular, the respective densities g of Z_0^* and h of σ under $\mathbb{N}_0^*$ are given by

$$g(z) = \sqrt{\frac{3}{2\pi}}\, z^{-5/2}, \quad z > 0,$$

and

$$h(s) = \frac{\sqrt{3}}{2\pi} 2^{-1/4} \Gamma(3/4) s^{-7/4}, \quad s > 0.$$

Proof. We fix $\lambda > 0$ and $\mu > 0$, and compute

$$\mathbb{N}_0^*\Big(\sigma \, \exp(-\lambda Z_0^* - \mu\sigma)\Big).$$

Recalling (3.33), and using Lemma 3.2.10, we get that $Z_0^*(\mathrm{tr}_{-b}(W)) = Z_b$, $\mathbb{N}_0$ a.e. on $\{Z_b > 0\}$, for every $b > 0$. Hence, by applying Theorem 3.5.1 to the function $G(\omega) = \exp(-\lambda Z_0^*(\omega) - \mu\sigma(\omega))$, we obtain

$$\mathbb{N}_0^*\Big(\sigma \, \exp(-\lambda Z_0^* - \mu\sigma)\Big) = 2\int_0^\infty \mathrm{d}b \, \mathbb{N}_0\Big(Z_b \exp(-\lambda Z_b - \mu \mathcal{Y}_b)\Big)$$

with the notation

$$\mathcal{Y}_b = \int_0^\sigma \mathrm{d}s \, \mathbf{1}_{\{\tau_{-b}(W_s) = \infty\}}$$

(note that $\mathcal{Y}_b = \sigma(\mathrm{tr}_{-b}(W))$, $\mathbb{N}_0$ a.e.). Set

$$u_{\lambda,\mu}(b) = \mathbb{N}_0\Big(1 - \exp(-\lambda Z_b - \mu \mathcal{Y}_b)\Big),$$

and note that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} u_{\lambda,\mu}(b) = \mathbb{N}_0\Big(Z_b \exp(-\lambda Z_b - \mu \mathcal{Y}_b)\Big).$$

The quantity $u_{\lambda,\mu}(b)$ is computed explicitly in [25, Lemma 4.5]: If $\lambda < \sqrt{\frac{\mu}{2}}$,

$$u_{\lambda,\mu}(b) = \sqrt{\frac{\mu}{2}} \left(3 \left(\tanh^2 \left((2\mu)^{1/4} b + \tanh^{-1} \sqrt{\frac{2}{3} + \frac{1}{3}\sqrt{\frac{2}{\mu}}\lambda} \right) \right) - 2 \right),$$

and a similar formula holds if $\lambda > \sqrt{\frac{\mu}{2}}$. From this explicit formula, one gets, say in the case $\lambda < \sqrt{\frac{\mu}{2}}$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\lambda} u_{\lambda,\mu}(b) = & K_{\lambda,\mu}^{-1} \tanh \left((2\mu)^{1/4} b + \tanh^{-1} \sqrt{\frac{2}{3} + \frac{1}{3}\sqrt{\frac{2}{\mu}}\lambda} \right) \\ & \times \left(\cosh^2 \left((2\mu)^{1/4} b + \tanh^{-1} \sqrt{\frac{2}{3} + \frac{1}{3}\sqrt{\frac{2}{\mu}}\lambda} \right) \right)^{-1} \end{aligned}$$

where

$$K_{\lambda,\mu} = \frac{1}{3}\left(1 - \sqrt{\frac{2}{\mu}}\lambda\right)\sqrt{\frac{2}{3} + \frac{1}{3}\sqrt{\frac{2}{\mu}}\lambda}.$$

By integrating the last formula between $b = 0$ and $b = \infty$, we arrive at

$$\int_0^\infty \mathrm{d}b\, \mathbb{N}_0\Big(Z_b \exp(-\lambda Z_b - \mu \mathcal{Y}_b)\Big) = \int_0^\infty \mathrm{d}b\, \frac{d}{d\lambda} u_{\lambda,\mu}(b) = \frac{1}{2}\sqrt{\frac{3}{2}}\Big(\lambda + \sqrt{2\mu}\Big)^{-1/2}.$$

Similar calculations give the same result when $\lambda > \sqrt{\frac{\mu}{2}}$ (and also in the case $\lambda = \sqrt{\frac{\mu}{2}}$ by a suitable passage to the limit). Summarizing, we have proved that, for every $\lambda > 0$ and $\mu > 0$,

$$\mathbb{N}_0^*\Big(\sigma\, \exp(-\lambda Z_0^* - \mu\sigma)\Big) = \sqrt{\frac{3}{2}}\Big(\lambda + \sqrt{2\mu}\Big)^{-1/2}.$$

At this stage, we only need to verify that, with the function f defined in the proposition, we have also

$$\int_0^\infty \int_0^\infty s \exp(-\lambda z - \mu s)\, f(z,s)\, \mathrm{d}z\mathrm{d}s = \sqrt{\frac{3}{2}}\Big(\lambda + \sqrt{2\mu}\Big)^{-1/2}.$$

To see this, first note that, for every $z > 0$,

$$z\int_0^\infty s^{-3/2} \exp\Big(-\frac{z^2}{2s} - \mu z\Big)\, \mathrm{d}s = \sqrt{2\pi}\, e^{-z\sqrt{2\mu}},$$

by the classical formula for the Laplace transform of a standard linear Brownian motion. The desired result easily follows. □

We now state a technical result that will be important for our purposes. Let us fix $\delta > 0$, and, for every $\varepsilon \in (0,\delta)$, write $W^{\delta,\varepsilon}$ for a random snake trajectory distributed according to $\mathbb{N}_\varepsilon(\cdot \mid \tilde{M} > \delta)$, where we recall the notation $\tilde{M} = \sup\{W_s(t) : s \geq 0, t \leq \zeta_s \wedge \tau_0(W_s)\}$. As usual, write $\tilde{W}^{\delta,\varepsilon}$ for $W^{\delta,\varepsilon}$ truncated at level 0. By Corollary 3.3.10, the distribution of $\tilde{W}^{\delta,\varepsilon}$ converges to $\mathbb{N}_0^*(\cdot \mid M > \delta)$ as $\varepsilon \to 0$. The next proposition shows that this convergence holds jointly with that of the exit measures from $(0,\infty)$. Recall the notation $\mathcal{Z}_0(W^{\delta,\varepsilon})$ for the (total mass of the) exit measure of $W^{\delta,\varepsilon}$ from $(0,\infty)$.

Proposition 3.6.3. *As $\varepsilon \to 0$, the distribution of the pair $(\tilde{W}^{\delta,\varepsilon}, \mathcal{Z}_0(W^{\delta,\varepsilon}))$ converges weakly to that of the pair $(W^{\delta,0}, Z_0^*(W^{\delta,0}))$, where $W^{\delta,0}$ is distributed according to $\mathbb{N}_0^*(\cdot \mid M > \delta)$.*

Proof. We may argue along a sequence $(\varepsilon_n)_{n\geq 1}$ strictly decreasing to 0. To simplify notation, we set $W^n = W^{\delta,\varepsilon_n}$ and $\tilde{W}^n = \tilde{W}^{\delta,\varepsilon_n}$. From Proposition 3.4.1, we may construct the whole sequence $(W^n)_{n\geq 1}$ and the snake trajectory $W^{\delta,0}$ in such a way that W^n is an excursion of W^m outside $(0,\varepsilon_n)$ for every $n < m$, $\tilde{W}^n$ is a subtrajectory of $W^{\delta,0}$ for every $n \geq 1$, and moreover $\sigma(\tilde{W}^n) \uparrow \sigma(W^{\delta,0})$ as $n \to \infty$. The latter properties imply that, for every $\gamma > 0$ and every $1 \leq n \leq m$, we have

$$\int_0^{\sigma(W^n)} \mathrm{d}s\, \mathbf{1}_{\{\zeta_s^n \leq \tau_0(W_s^n),\, \hat{W}_s^n < \gamma\}} \leq \int_0^{\sigma(W^m)} \mathrm{d}s\, \mathbf{1}_{\{\zeta_s^m \leq \tau_0(W_s^m),\, \hat{W}_s^m < \gamma\}} \leq \int_0^{\sigma(W^{\delta,0})} \mathrm{d}s\, \mathbf{1}_{\{\hat{W}_s^{\delta,0} < \gamma\}}.$$

If we multiply this inequality by γ^{-2} and let γ tend to 0, we obtain that, for every $1 \leq n \leq m$,

$$\mathcal{Z}_0(W^n) \leq \mathcal{Z}_0(W^m) \leq Z_0^*(W^{\delta,0}).$$

In particular the almost sure increasing limit

$$Z_0' := \lim_{n\to\infty} \uparrow \mathcal{Z}_0(W^n)$$

exists and we have $Z'_0 \leq Z^*_0(W^{\delta,0})$. The result of the proposition will follow if we can verify that we have indeed $Z'_0 = Z^*_0(W^{\delta,0})$. To this end, fix $\lambda > 0$ and $\mu > 0$. Note that

$$\mathbb{N}^*_0[\exp(-\lambda Z'_0)(1-\exp(-\mu\sigma(W^{\delta,0})))] \leq \liminf_{n\to\infty} \mathbb{N}_{\varepsilon_n} \exp(-\lambda \mathcal{Z}_0(W^n))(1-\exp(-\mu\sigma(\tilde{W}^n)))] \quad (3.34)$$

by Fatou's lemma. We will verify that

$$\liminf_{n\to\infty} \mathbb{N}_{\varepsilon_n}[\exp(-\lambda \mathcal{Z}_0(W^n))(1-\exp(-\mu\sigma(\tilde{W}^n)))] \leq \mathbb{N}^*_0[\exp(-\lambda Z^*_0(W^{\delta,0}))(1-\exp(-\mu\sigma(W^{\delta,0})))]. \quad (3.35)$$

If (3.35) holds, then by combining this with the previous display, we get

$$\mathbb{N}^*_0[\exp(-\lambda Z'_0)(1-\exp(-\mu\sigma(W^{\delta,0})))] \leq \mathbb{N}^*_0[\exp(-\lambda Z^*_0(W^{\delta,0}))(1-\exp(-\mu\sigma(W^{\delta,0})))],$$

and since we already know that $Z'_0 \leq Z^*_0(W^{\delta,0})$, this is only possible if $Z'_0 = Z^*_0(W^{\delta,0})$.

Let us prove (3.35). Recalling that $\mathbb{N}_\varepsilon(\tilde{M} > \delta) \sim \varepsilon\, \mathbb{N}^*_0(M > \delta)$ as $\varepsilon \to 0$, we see that (3.35) is equivalent to

$$\liminf_{n\to\infty} \frac{1}{\varepsilon_n} \mathbb{N}_{\varepsilon_n}\Big(\exp(-\lambda \mathcal{Z}_0)(1-\exp(-\mu \mathcal{Y}_0))\,\mathbf{1}_{\{\tilde{M}>\delta\}}\Big) \leq \mathbb{N}^*_0(\exp(-\lambda Z^*_0)(1-\exp(-\mu\sigma))\,\mathbf{1}_{\{M>\delta\}}), \quad (3.36)$$

where we recall that

$$\mathcal{Y}_0 = \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\tau_0(W_s)=\infty\}}.$$

Observe that, for any choice of $\gamma \in (0,\delta)$, the argument leading to (3.34) also gives

$$\liminf_{n\to\infty} \frac{1}{\varepsilon_n} \mathbb{N}_{\varepsilon_n}\Big(\exp(-\lambda \mathcal{Z}_0)(1-\exp(-\mu \mathcal{Y}_0))\,\mathbf{1}_{\{\gamma<\tilde{M}\leq\delta\}}\Big) \geq \mathbb{N}^*_0(\exp(-\lambda Z^*_0)(1-\exp(-\mu\sigma))\,\mathbf{1}_{\{\gamma<M\leq\delta\}}). \quad (3.37)$$

and by letting γ tend to 0,

$$\liminf_{n\to\infty} \frac{1}{\varepsilon_n} \mathbb{N}_{\varepsilon_n}\Big(\exp(-\lambda \mathcal{Z}_0)(1-\exp(-\mu \mathcal{Y}_0))\,\mathbf{1}_{\{\tilde{M}\leq\delta\}}\Big) \geq \mathbb{N}^*_0(\exp(-\lambda Z^*_0)(1-\exp(-\mu\sigma))\,\mathbf{1}_{\{M\leq\delta\}}).$$

So if (3.36) fails, we get

$$\liminf_{n\to\infty} \frac{1}{\varepsilon_n} \mathbb{N}_{\varepsilon_n}\Big(\exp(-\lambda \mathcal{Z}_0)(1-\exp(-\mu \mathcal{Y}_0))\Big) > \mathbb{N}^*_0(\exp(-\lambda Z^*_0)(1-\exp(-\mu\sigma))).$$

We will prove that we have

$$\lim_{n\to\infty} \frac{1}{\varepsilon_n} \mathbb{N}_{\varepsilon_n}\Big(\exp(-\lambda \mathcal{Z}_0)(1-\exp(-\mu \mathcal{Y}_0))\Big) = \mathbb{N}^*_0(\exp(-\lambda Z^*_0)(1-\exp(-\mu\sigma))), \quad (3.38)$$

showing by contradiction that (3.36) and thus also (3.35) hold.

The right-hand side of (3.38) can be computed from the formula

$$\mathbb{N}^*_0(\sigma\exp(-\lambda Z^*_0 - \mu\sigma)) = \sqrt{\frac{3}{2}}\Big(\lambda + \sqrt{2\mu}\Big)^{-1/2},$$

which was obtained in the proof of Proposition 3.6.2. We get

$$\begin{aligned}\mathbb{N}_0^*(\exp(-\lambda Z_0^*)(1-\exp(-\mu\sigma))) &= \mathbb{N}_0^*\Big(\exp(-\lambda Z_0^*)\int_0^\mu \mathrm{d}\mu'\,\sigma\exp(-\mu'\sigma)\Big) \\ &= \int_0^\mu \mathrm{d}\mu'\,\sqrt{\frac{3}{2}}\Big(\lambda+\sqrt{2\mu'}\Big)^{-1/2} \\ &= \sqrt{\frac{3}{2}}\int_0^{\sqrt{2\mu}} \mathrm{d}x\,x\,(\lambda+x)^{-1/2} \\ &= \sqrt{\frac{2}{3}}\Big((\lambda+\sqrt{2\mu})^{3/2}-3\lambda\,(\lambda+\sqrt{2\mu})^{1/2}+2\lambda^{3/2}\Big).\end{aligned} \tag{3.39}$$

On the other hand, we have, for every $\varepsilon>0$,

$$\begin{aligned}\mathbb{N}_\varepsilon\Big(\exp(-\lambda\mathcal{Z}_0)(1-\exp(-\mu\mathcal{Y}_0))\Big) &= \mathbb{N}_\varepsilon\Big(1-\exp(-\lambda\mathcal{Z}_0-\mu\mathcal{Y}_0)\Big)-\mathbb{N}_\varepsilon\Big(1-\exp(-\lambda\mathcal{Z}_0)\Big) \\ &= u_{\lambda,\mu}(\varepsilon)-\Big(\frac{1}{\sqrt{\lambda}}+\varepsilon\sqrt{\frac{2}{3}}\Big)^{-2},\end{aligned}$$

with the notation introduced in the proof of Proposition 3.6.2. Formula (26) in [25] gives

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\,(u_{\lambda,\mu}(\varepsilon)-\lambda)=\frac{\mathrm{d}}{\mathrm{d}\varepsilon}u_{\lambda,\mu}(\varepsilon)_{|\varepsilon=0}=\sqrt{\frac{2}{3}}\,(\lambda+\sqrt{2\mu})^{1/2}\,(\sqrt{2\mu}-2\lambda).$$

It follows that

$$\lim_{n\to\infty}\frac{1}{\varepsilon_n}\,\mathbb{N}_{\varepsilon_n}\Big(\exp(-\lambda\mathcal{Z}_0)(1-\exp(-\mu\mathcal{Y}_0))\Big)=\sqrt{\frac{2}{3}}\,(\lambda+\sqrt{2\mu})^{1/2}\,(\sqrt{2\mu}-2\lambda)+2\sqrt{\frac{2}{3}}\,\lambda^{3/2},$$

and one immediately verifies that the right-hand side coincides with the right-hand side of (3.39). This completes the proof of (3.38) and of the proposition. □

In view of our applications, it will be important to define the measure $\mathbb{N}_0^*$ conditioned on a given value of the exit measure. This is the goal of the next proposition. Before that, we mention a useful scaling property. Recall the definition of the scaling operator θ_λ at the end of Section 3.3. Then for every $\lambda>0$, we have for every $\omega\in\mathcal{S}$,

$$Z_0^*\circ\theta_\lambda(\omega)=\lambda\,Z_0^*(\omega). \tag{3.40}$$

The proof is easy, recalling from (3.33) the definition of $Z_0^*(\omega)$ for an arbitrary $\omega\in\mathcal{S}$ and writing

$$\begin{aligned}Z_0^*\circ\theta_\lambda(\omega) &= \liminf_{n\to\infty}\alpha_n^{-2}\int_0^{\lambda^2\sigma(\omega)}\mathrm{d}s\,\mathbf{1}_{\{\sqrt{\lambda}\hat W_{s/\lambda^2}(\omega)<\sqrt{\lambda}W_*+\alpha_n\}} \\ &= \lambda\liminf_{n\to\infty}(\alpha_n/\sqrt{\lambda})^{-2}\int_0^{\sigma(\omega)}\mathrm{d}s\,\mathbf{1}_{\{\hat W_s<W_*+\alpha_n/\sqrt{\lambda}\}} \\ &= \lambda Z_0^*(\omega).\end{aligned}$$

Proposition 3.6.4. *There exists a unique collection $(\mathbb{N}_0^{*,z})_{z>0}$ of probability measures on $\mathcal{S}$, which depends continuously on z, such that:*

(i) *We have*

$$\mathbb{N}_0^* = \sqrt{\frac{3}{2\pi}} \int_0^\infty \mathrm{d}z\, z^{-5/2}\, \mathbb{N}_0^{*,z}.$$

(ii) *For every $z > 0$, $\mathbb{N}_0^{*,z}$ is supported on $\{Z_0^* = z\}$.*

(iii) *For every $z, z' > 0$, $\mathbb{N}_0^{*,z'} = \theta_{z'/z}(\mathbb{N}_0^{*,z})$.*

We will write $\mathbb{N}_0^{*,z} = \mathbb{N}_0^*(\cdot \mid Z_0^* = z)$.

Proof. Recall from Proposition 3.6.2 that the "law" of Z_0^* under $\mathbb{N}_0^*$ is given by the measure $\mathbf{1}_{\{z>0\}} \sqrt{3/2\pi}\, z^{-5/2}\, \mathrm{d}z$, which we denote here by $\nu(\mathrm{d}z)$ to simplify notation. The existence of a collection of probability measures on $\mathcal{S}$ that satisfy both (i) and (ii) in the proposition is a consequence of standard desintegration theorems (see e.g. [26, Chapter III]). Two such collections coincide up a negligible set of values of z. We need to verify that we can choose this collection so that the additional scaling property (iii) also holds (which will imply the stronger uniqueness in the proposition).

We start with any measurable collection $(\mathbb{Q}_z)_{z>0}$ of probability measures on $\mathcal{S}$ such that the properties stated in (i) and (ii) hold when $(\mathbb{N}_0^{*,z})_{z>0}$ is replaced by $(\mathbb{Q}_z)_{z>0}$. From Lemma 3.3.11, we get that, for every $\lambda > 0$,

$$\int \theta_\lambda(\mathbb{Q}_z)\, \nu(\mathrm{d}z) = \theta_\lambda(\mathbb{N}_0^*) = \lambda^{3/2}\mathbb{N}_0^* = \lambda^{3/2} \int \mathbb{Q}_z\, \nu(\mathrm{d}z).$$

From the change of variables $z = z'/\lambda$ in the first integral, we thus get

$$\int \theta_\lambda(\mathbb{Q}_{z/\lambda})\, \nu(\mathrm{d}z) = \int \mathbb{Q}_z\, \nu(\mathrm{d}z).$$

Using the scaling property (3.40), we see that the collection $(\theta_\lambda(\mathbb{Q}_{z/\lambda}))_{z>0}$ also satisfy the conditions (i) and (ii), and so we get for every fixed $\lambda > 0$,

$$\theta_\lambda(\mathbb{Q}_{z/\lambda}) = \mathbb{Q}_z\,, \quad \mathrm{d}z \text{ a.e.}$$

From Fubini's theorem, we have then $\theta_\lambda(\mathbb{Q}_{z/\lambda}) = \mathbb{Q}_z$, $\mathrm{d}\lambda$ a.e., $\mathrm{d}z$ a.e. At this stage, we can pick $z_0 > 0$ such that the equality $\theta_\lambda(\mathbb{Q}_{z_0/\lambda}) = \mathbb{Q}_{z_0}$ holds $\mathrm{d}\lambda$ a.e., and define $\mathbb{N}_0^{*,z} := \theta_{z/z_0}(\mathbb{Q}_{z_0})$ for every $z > 0$. We have then $\mathbb{N}_0^{*,z} = \mathbb{Q}_z$, $\mathrm{d}z$ a.e., so that (i) holds for the collection $(\mathbb{N}_0^{*,z})_{z>0}$. Similarly (ii) holds because $\mathbb{Q}_{z_0}$ is supported on $\{Z_0^* = z_0\}$, and we use the scaling property (3.40). Finally (iii) holds by construction, and implies the continuous dependence of $(\mathbb{N}_0^{*,z})_{z>0}$ in the variable z. The uniqueness is then a simple consequence of this continuous dependence. ☐

3.7 The excursion process

For technical reasons in this section, it is preferable to argue under a probability measure rather than under $\mathbb{N}_0$. So we fix $\beta > 0$, and we argue under the conditional measure $\mathbb{N}_0^{(\beta)} := \mathbb{N}_0(\cdot \mid W_* < -\beta)$. We will then consider, under $\mathbb{N}_0^{(\beta)}$, the excursion debuts whose level is smaller than $-\beta$. For every $\delta > 0$, we write $u_1^\delta, \ldots, u_{N_\delta}^\delta$ for the excursion debuts with height greater than δ whose level is smaller than $-\beta$, listed in decreasing order of the levels, so that

$$V_{u^\delta_{N_\delta}} < V_{u^\delta_{N_\delta - 1}} < \cdots < V_{u^\delta_1} < -\beta.$$

Notice that N_δ and $u_1^\delta,\ldots,u_{N_\delta}^\delta$ depend on the choice of β, which will remain fixed in the first three subsections below (although on a couple of occasions we mention the applications that one derives by letting β tend to 0, but this should create no confusion). For every integer $i\geq 1$, we also set

$$T_i^\delta := \begin{cases} -V_{u_i^\delta} & \text{if } i\leq N_\delta,\\ \infty & \text{if } i>N_\delta.\end{cases}$$

It is easy to verify that, for every $a>0$, the event $\{T_i^\delta<a\}$ belongs to the σ-field $\mathcal{E}^{(-a,\infty)}$ (the knowledge of $\mathcal{E}^{(-a,\infty)}$ gives enough information to recover the excursion debuts – and the corresponding heights – such that $V_u>-a$). Since $\{T_i^\delta=a\}$ is $\mathbb{N}_0$-negligible, it follows that T_i^δ is a stopping time of the filtration $(\mathcal{E}^{(-a,\infty)})_{a\geq 0}$, where, by convention, $\mathcal{E}^{(0,\infty)}$ is the σ-field generated by the $\mathbb{N}_0$-negligible sets. Finally, it will also be useful to write $N_\delta^\circ=\#D_\delta$ for the total number of excursion debuts with height greater than δ.

3.7.1 The excursions with height greater than δ

Recall the notation $\tilde W^{(u)}$ for the excursion starting at the excursion debut $u\in D$.

Proposition 3.7.1. *Let $j\geq 1$. Then, under the conditional probability measure $\mathbb{N}_0^{(\beta)}(\cdot\mid N_\delta\geq j)$, $\tilde W^{(u_j^\delta)}$ is independent of the σ-field generated by $(\tilde W^{(u_1^\delta)},\ldots,\tilde W^{(u_{j-1}^\delta)})$ and $\mathcal{E}^{(-\beta,\infty)}$, and is distributed according to $\mathbb{N}_0^*(\cdot\mid M>\delta)$.*

Important remark. In view of the analogous statement for linear Brownian motion, one might naively expect that $\tilde W^{(u_1^\delta)},\ldots,\tilde W^{(u_j^\delta)}$ are (independent and) identically distributed under $\mathbb{N}_0^{(\beta)}(\cdot\mid N_\delta\geq j)$. This is not true as soon as $j\geq 2$: The point is that the knowledge of the event $\{N_\delta\geq j\}$ influences the distribution of $(\tilde W^{(u_1^\delta)},\ldots,\tilde W^{(u_{j-1}^\delta)})$.

Proof. The first step of the proof is to determine the law of $\tilde W^{(u_1^\delta)}$ under $\mathbb{N}_0^{(\beta)}(\cdot\mid N_\delta\geq 1)$. We fix two bounded nonnegative functions G and g defined respectively on $\mathcal{S}$ and on $\mathbb{R}$. We assume that G is bounded and continuous on the set $\{\omega: M(\omega)>\delta\}$, and vanishes outside this set. The function g is assumed to be continuous with compact support contained in $(-\infty,-\beta]$.

We retain much of the notation of the proof of Theorem 3.3.7. In particular, for every integers $n\geq 1$ and $k\geq 1$, we let $\mathcal{N}_k^{2^{-n}}$ be the point measure of excursions of the Brownian snake outside $(-k2^{-n},\infty)$, and we write

$$\mathcal{N}_k^{2^{-n}}=\sum_{i\in I_k^{2^{-n}}}\delta_{\omega_i^{k,2^{-n}}}.$$

For every atom $\omega_i^{k,2^{-n}}$, we use the notation $\tilde\omega_i^{k,2^{-n}}$ for $\omega_i^{k,2^{-n}}$ truncated at level $-(k+1)2^{-n}$ and translated so that its starting point is 2^{-n}. Furthermore, we let $A_{n,k}$ stand for the event $\{T_1^\delta\geq k2^{-n}\}$. Finally, we let $B\in\mathcal{E}^{(-\beta,\infty)}$.

We then claim that

$$\mathbb{N}_0^{(\beta)}\Big(\mathbf{1}_B\,g(V_{u_1^\delta})\,G(\tilde W^{(u_1^\delta)})\,\mathbf{1}_{\{N_\delta\geq 1\}}\Big)=\lim_{n\to\infty}\mathbb{N}_0^{(\beta)}\left(\mathbf{1}_B\sum_{k=1}^{\infty}\mathbf{1}_{A_{n,k}}g(-k2^{-n})\sum_{i\in I_k^{2^{-n}}}G(\tilde\omega_i^{k,2^{-n}})\right),\quad(3.41)$$

In order to verify our claim, we first observe that

$$\sum_{k=1}^{\infty} \mathbf{1}_{A_{n,k}} g(-k2^{-n}) \sum_{i\in I_k^{2^{-n}}} G(\tilde{\omega}_i^{k,2^{-n}}) \underset{n\to\infty}{\longrightarrow} g(V_{u_1^\delta})\, G(\tilde{W}^{(u_1^\delta)})\, \mathbf{1}_{\{N_\delta\geq 1\}}, \qquad \mathbb{N}_0 \text{ a.e.} \tag{3.42}$$

To see this, note that if $N_\delta = 0$ then, for n large enough, all quantities $G(\tilde{\omega}_i^{k,2^{-n}})$ vanish (the point is that, if $G(\tilde{\omega}_i^{k,2^{-n}}) > 0$, then the excursion $\omega_i^{k,2^{-n}}$ must "contain" an excursion debut with height greater than $\delta - 2^{-n}$, and no such excursion debut exists when n is large enough, under the condition $N_\delta = 0$). Then, if $N_\delta \geq 1$, similar arguments show that, for n large enough, the only nonzero term in the sum over k in the left-hand side of (3.42) corresponds to the integer k_0 such that $-(k_0+1)2^{-n} < V_{u_1^\delta} \leq -k_0 2^{-n}$ (observe that $\mathbf{1}_{A_{n,k}} = 0$ if $k > k_0$ and that the quantities $G(\tilde{\omega}_i^{k,2^{-n}})$ vanish if $k < k_0$ by the assumption on G). Furthermore, for $k = k_0$, the sum over $i \in I_k^{2^{-n}}$ reduces (for n large enough) to a single term, namely $i = i_0 = i_{u_\delta^1, 2^{-n}}$ with the notation of Lemma 3.3.8. The last assertion of Lemma 3.3.8 yields that $G(\tilde{\omega}_{i_0}^{k_0,2^{-n}})$ converges to $G(\tilde{W}^{(u_1^\delta)})$ as $n\to\infty$, and (3.42) follows.

To derive (3.41) from (3.42), we use exactly the same uniform integrability argument as in the proof of Theorem 3.3.7 to justify the convergence (3.20).

Next recall that $A_{n,k}$ is measurable with respect to the σ-field $\mathcal{E}^{(-k2^{-n},\infty)}$, and note that $g(-k2^{-n}) = 0$ if $k \leq 2^n\beta$. By applying the special Markov property, we then get

$$\begin{aligned}
&\mathbb{N}_0^{(\beta)}\Bigg(\mathbf{1}_B \sum_{k\geq 2^n\beta} \mathbf{1}_{A_{n,k}} g(-k2^{-n}) \sum_{i\in I_k^{2^{-n}}} G(\tilde{\omega}_i^{k,2^{-n}})\Bigg)\\
&\quad= \sum_{k\geq 2^n\beta} g(-k2^{-n})\, \mathbb{N}_0^{(\beta)}\Bigg(\mathbf{1}_B \mathbf{1}_{A_{n,k}} \mathbb{N}_0^{(\beta)}\Bigg(\sum_{i\in I_k^{2^{-n}}} G(\tilde{\omega}_i^{k,2^{-n}})\,\Bigg|\,\mathcal{E}^{(-k2^{-n},\infty)}\Bigg)\Bigg)\\
&\quad= \sum_{k\geq 2^n\beta} g(-k2^{-n})\, \mathbb{N}_0^{(\beta)}\Big(\mathbf{1}_B \mathbf{1}_{A_{n,k}}\, Z_{k2^{-n}}\, \mathbb{N}_{2^{-n}}\big(G(\tilde{W})\big)\Big)\\
&\quad= \Bigg(\sum_{k\geq 2^n\beta} g(-k2^{-n})\, \mathbb{N}_0^{(\beta)}\Big(\mathbf{1}_B \mathbf{1}_{A_{n,k}}\, Z_{k2^{-n}}\Big)\Bigg) \times \mathbb{N}_{2^{-n}}\big(G(\tilde{W})\big).
\end{aligned}$$

Recalling (3.42), we have thus obtained

$$\lim_{n\to\infty}\Bigg(\sum_{k\geq 2^n\beta} g(-k2^{-n}) \mathbb{N}_0^{(\beta)}\Big(\mathbf{1}_B \mathbf{1}_{A_{n,k}}\, Z_{k2^{-n}}\Big)\Bigg) \times \mathbb{N}_{2^{-n}}\big(G(\tilde{W})\big) = \mathbb{N}_0^{(\beta)}\Big(\mathbf{1}_B\, g(V_{u_1^\delta})\, G(\tilde{W}^{(u_1^\delta)})\, \mathbf{1}_{\{N_\delta\geq 1\}}\Big). \tag{3.43}$$

In the particular case $G = \mathbf{1}_{\{M>\delta\}}$ this gives

$$\lim_{n\to\infty}\Bigg(\sum_{k\geq 2^n\beta} g(-k2^{-n}) \mathbb{N}_0^{(\beta)}\Big(\mathbf{1}_B \mathbf{1}_{A_{n,k}}\, Z_{k2^{-n}}\Big)\Bigg) \times \mathbb{N}_{2^{-n}}\big(\tilde{M} > \delta\big) = \mathbb{N}_0^{(\beta)}\Big(\mathbf{1}_B\, g(V_{u_1^\delta})\, \mathbf{1}_{\{N_\delta\geq 1\}}\Big), \tag{3.44}$$

since $M(\tilde{W}^{(u_1^\delta)}) > \delta$ by construction. It follows from (3.43) and (3.44) that

$$\begin{aligned}\mathbb{N}_0^{(\beta)}\Big(\mathbf{1}_B\, g(V_{u_1^\delta})\, G(\tilde{W}^{(u_1^\delta)})\, \mathbf{1}_{\{N_\delta \geq 1\}}\Big) &= \mathbb{N}_0^{(\beta)}\Big(\mathbf{1}_B\, g(V_{u_1^\delta})\, \mathbf{1}_{\{N_\delta \geq 1\}}\Big) \times \lim_{n\to\infty} \frac{\mathbb{N}_{2^{-n}}(G(\tilde{W}))}{\mathbb{N}_{2^{-n}}(\tilde{M} > \delta)} \\ &= \mathbb{N}_0^{(\beta)}\Big(\mathbf{1}_B\, g(V_{u_1^\delta})\, \mathbf{1}_{\{N_\delta \geq 1\}}\Big) \times \mathbb{N}_0^*(G \mid M > \delta),\end{aligned}$$

by Corollary 3.3.10. The last display shows both that $\tilde{W}^{(u_1^\delta)}$ is distributed according to $\mathbb{N}_0^*(\cdot \mid M > \delta)$ under $\mathbb{N}_0^{(\beta)}(\cdot \mid N_\delta \geq 1)$ (take a sequence of functions g that increase to the indicator function of $(-\infty, -\beta)$) and that $\tilde{W}^{(u_1^\delta)}$ is independent of the σ-field generated by $V_{u_1^\delta}$ and $\mathcal{E}^{(\beta,\infty)}$, still under $\mathbb{N}_0^{(\beta)}(\cdot \mid N_\delta \geq 1)$.

We have obtained that the law of the first excursion above the minimum with height greater than δ and level smaller than $-\beta$, under $\mathbb{N}_0^{(\beta)}(\cdot \mid N_\delta \geq 1)$, is $\mathbb{N}_0^*(\cdot \mid M > \delta)$. By letting β tend to 0, we deduce that the law of the the first excursion above the minimum with height greater than δ, under $\mathbb{N}_0(\cdot \mid N_\delta^\circ \geq 1)$, is also $\mathbb{N}_0^*(\cdot \mid M > \delta)$ – we recall our notation N_δ° for the total number of excursion debuts with height greater than δ. Moreover, the same passage to the limit shows that this first excursion is independent of the level at which it occurs. These remarks will be useful in the second part of the proof.

The general statement of the proposition can be deduced from the special case $j = 1$, via an induction argument using the special Markov property. Let us explain this argument in detail when $j = 2$ (the reader will be able to fill in the details needed for a general value of j). Let G_1 and G_2 be two nonnegative measurable functions on $\mathcal{S}$, and consider again $B \in \mathcal{E}^{(-\beta,\infty)}$. Recall that $T_1^\delta > \beta$ by definition. By monotone convergence, we have

$$\begin{aligned}&\mathbb{N}_0^{(\beta)}\Big(\mathbf{1}_B\, G_1(\tilde{W}^{(u_1^\delta)})\, G_2(\tilde{W}^{(u_2^\delta)})\, \mathbf{1}_{\{N_\delta \geq 2\}}\Big) \\ &\quad= \lim_{n\to\infty} \sum_{k \geq 2^n\beta} \mathbb{N}_0^{(\beta)}\Big(\mathbf{1}_B\, G_1(\tilde{W}^{(u_1^\delta)})\, G_2(\tilde{W}^{(u_2^\delta)})\, \mathbf{1}_{\{k2^{-n} \leq T_1^\delta < (k+1)2^{-n} \leq T_2^\delta < \infty\}}\Big).\end{aligned} \tag{3.45}$$

Then, noting that $\mathbf{1}_B\, G_1(\tilde{W}^{(u_1^\delta)})\, \mathbf{1}_{\{T_1^\delta < (k+1)2^{-n}\}}$ is $\mathcal{E}^{(-(k+1)2^{-n},\infty)}$-measurable, we get, for every $k \geq 2^n\beta$,

$$\begin{aligned}&\mathbb{N}_0^{(\beta)}\Big(\mathbf{1}_B\, G_1(\tilde{W}^{(u_1^\delta)})\, G_2(\tilde{W}^{(u_2^\delta)})\, \mathbf{1}_{\{k2^{-n} \leq T_1^\delta < (k+1)2^{-n} \leq T_2^\delta < \infty\}}\Big) \\ &\quad= \mathbb{N}_0^{(\beta)}\Big(\mathbf{1}_B\, G_1(\tilde{W}^{(u_1^\delta)})\, \mathbf{1}_{\{k2^{-n} \leq T_1^\delta < (k+1)2^{-n} \leq T_2^\delta\}}\, \mathbb{N}_0^{(\beta)}\Big(G_2(\tilde{W}^{(u_2^\delta)})\, \mathbf{1}_{\{T_2^\delta < \infty\}} \,\Big|\, \mathcal{E}^{(-(k+1)2^{-n},\infty)}\Big)\Big).\end{aligned}$$

Applying the special Markov property to the domain $(-(k+1)2^{-n}, \infty)$ now gives on the event $\{T_1^\delta < (k+1)2^{-n} \leq T_2^\delta\}$,

$$\mathbb{N}_0^{(\beta)}\Big(G_2(\tilde{W}^{(u_2^\delta)})\, \mathbf{1}_{\{T_2^\delta < \infty\}} \,\Big|\, \mathcal{E}^{(-(k+1)2^{-n},\infty)}\Big) = \Big(1 - \exp(-Z_{(k+1)2^{-n}}\, \mathbb{N}_0(N_\delta \geq 1))\Big)\, \mathbb{N}_0^*(G_2 \mid M > \delta). \tag{3.46}$$

Let us explain this. From the special Markov property, there is a Poisson number ν with parameter $Z_{(k+1)2^{-n}}\mathbb{N}_0(N_\delta^\circ \geq 1)$ of Brownian snake excursions outside $(-(k+1)2^{-n}, \infty)$ that contain at least one excursion debut with height greater than δ, and these excursions are independent and distributed according to $\mathbb{N}_0(\cdot \mid N_\delta^\circ \geq 1)$, modulo the obvious translation by $(k+1)2^{-n}$. For each of these ν excursions, the first excursion above the minimum with height greater than δ is distributed according to $\mathbb{N}_0^*(\cdot \mid M > \delta)$, and is independent of the level at which it occurs (by the first part of the proof). On the event $\{T_1^\delta < (k+1)2^{-n} \leq T_2^\delta\}$, $\tilde{W}^{(u_2^\delta)}$ is well defined if $T_2^\delta < \infty$, which is equivalent to $\nu \geq 1$, and is obtained by taking among these first excursions above the mimimum with height greater than δ the one that occurs at the highest level. Clearly it is also distributed according to $\mathbb{N}_0^*(\cdot \mid M > \delta)$.

Since $1 - \exp(-Z_{(k+1)2^{-n}}\,\mathbb{N}_0(N_\delta^\circ \geq 1)) = \mathbb{N}_0^{(\beta)}(T_2^\delta < \infty \mid \mathcal{E}^{(-(k+1)2^{-n},\infty)})$ on the event $\{T_1^\delta < (k+1)2^{-n} \leq T_2^\delta\}$, we deduce from (3.46) that, for every $k \geq 2^n\beta$,

$$\begin{aligned}
&\mathbb{N}_0^{(\beta)}\Big(\mathbf{1}_B\, G_1(\tilde{W}^{(u_1^\delta)})\, G_2(\tilde{W}^{(u_2^\delta)})\,\mathbf{1}_{\{k2^{-n}\leq T_1^\delta<(k+1)2^{-n}\leq T_2^\delta<\infty\}}\Big)\\
&\quad= \mathbb{N}_0^{(\beta)}\Big(\mathbf{1}_B\, G_1(\tilde{W}^{(u_1^\delta)})\,\mathbf{1}_{\{k2^{-n}\leq T_1^\delta<(k+1)2^{-n}\leq T_2^\delta\}}\,\mathbb{N}_0^{(\beta)}(T_2^\delta < \infty \mid \mathcal{E}^{(-(k+1)2^{-n},\infty)})\Big) \times \mathbb{N}_0^*(G_2 \mid M > \delta)\\
&\quad= \mathbb{N}_0^{(\beta)}\Big(\mathbf{1}_B\, G_1(\tilde{W}^{(u_1^\delta)})\,\mathbf{1}_{\{k2^{-n}\leq T_1^\delta<(k+1)2^{-n}\leq T_2^\delta<\infty\}}\Big) \times \mathbb{N}_0^*(G_2 \mid M > \delta)
\end{aligned}$$

Finally, returning to (3.45), we obtain by monotone convergence

$$\mathbb{N}_0^{(\beta)}\Big(\mathbf{1}_B\, G_1(\tilde{W}^{(u_1^\delta)})\, G_2(\tilde{W}^{(u_2^\delta)})\,\mathbf{1}_{\{N_\delta\geq 2\}}\Big) = \mathbb{N}_0^{(\beta)}\Big(\mathbf{1}_B\, G_1(\tilde{W}^{(u_1^\delta)})\,\mathbf{1}_{\{N_\delta\geq 2\}}\Big)\,\mathbb{N}_0^*(G_2 \mid M > \delta).$$

This gives the case $j = 2$ of the proposition. □

Remark. We could have shortened the proof a little by using a strong version of the special Markov property (applying to a random interval $(-T,\infty)$) of the type discussed in [25].

The next lemma shows that the sequence $(\tilde{W}^{(u_1^\delta)}, \ldots, \tilde{W}^{(u_{N_\delta}^\delta)})$ can be viewed as the beginning of an i.i.d. sequence.

Lemma 3.7.2. *On an auxiliary probability space $(\Omega, \mathcal{F}, \mathbb{P})$, consider a sequence $(\overline{W}^{\delta,1}, \overline{W}^{\delta,2}, \ldots)$ of independent random variables distributed according to $\mathbb{N}_0^*(\cdot \mid M > \delta)$. Under the product probability measure $\mathbb{P} \otimes \mathbb{N}_0^{(\beta)}$, consider the sequence $(W^{\delta,1}, W^{\delta,2}, \ldots)$ defined by*

$$W^{\delta,j} = \begin{cases} \tilde{W}^{(u_j^\delta)} & \text{if } 1 \leq j \leq N_\delta \\ \overline{W}^{\delta,j-N_\delta} & \text{if } j > N_\delta \end{cases}$$

Then $(W^{\delta,1}, W^{\delta,2}, \ldots)$ is a sequence of i.i.d. random variables distributed according to $\mathbb{N}_0^*(\cdot \mid M > \delta)$, and this sequence is independent of the σ-field $\mathcal{E}^{(-\beta,\infty)}$.

Proof. This follows from Proposition 3.7.1 by an argument which is valid in a much more general setting. Let us give a few details. Let $k \geq 2$, and let $\phi_1, \ldots, \phi_k$ be bounded nonnegative measurable functions defined on $\mathcal{S}$. Also let $B \in \mathcal{E}^{(-\beta,\infty)}$. We need to verify that

$$E\Big[\mathbf{1}_B\,\phi_1(W^{\delta,1})\,\phi_2(W^{\delta,2})\cdots\phi_k(W^{\delta,k})\Big] = \mathbb{N}_0^{(\beta)}(B) \times \prod_{i=1}^k \mathbb{N}_0^*(\phi_i \mid M > \delta), \tag{3.47}$$

where $E[\cdot]$ stands for the expectation under $\mathbb{P}\otimes\mathbb{N}_0^{(\beta)}$. By dealing separately with the possible values of N_δ and using the independence of the $\overline{W}^{\delta,j}$'s, we immediately get that

$$\begin{aligned}
&E\Big[\mathbf{1}_{\{N_\delta<k\}}\mathbf{1}_B\,\phi_1(W^{\delta,1})\,\phi_2(W^{\delta,2})\cdots\phi_k(W^{\delta,k})\Big]\\
&\qquad= E\Big[\mathbf{1}_{\{N_\delta<k\}}\mathbf{1}_B\,\phi_1(W^{\delta,1})\cdots\phi_{k-1}(W^{\delta,k-1})\Big] \times \mathbb{N}_0^*(\phi_k \mid M > \delta).
\end{aligned}$$

On the other hand, Proposition 3.7.1 exactly says that

$$\begin{aligned}E\Big[&\mathbf{1}_{\{N_\delta\geq k\}}\mathbf{1}_B\,\phi_1(W^{\delta,1})\,\phi_2(W^{\delta,2})\cdots\phi_k(W^{\delta,k})\Big]\\&=E\Big[\mathbf{1}_{\{N_\delta\geq k\}}\mathbf{1}_B\,\phi_1(\tilde W^{(u_1^\delta)})\,\phi_2(\tilde W^{(u_2^\delta)})\cdots\phi_k(\tilde W^{(u_k^\delta)})\Big]\\&=E\Big[\mathbf{1}_{\{N_\delta\geq k\}}\mathbf{1}_B\,\phi_1(W^{\delta,1})\cdots\phi_{k-1}(W^{\delta,k-1})\Big]\times\mathbb{N}_0^*(\phi_k\mid M>\delta).\end{aligned}$$

By summing the last two displays, we get

$$E\Big[\mathbf{1}_B\,\phi_1(W^{\delta,1})\,\phi_2(W^{\delta,2})\cdots\phi_k(W^{\delta,k})\Big]=E\Big[\mathbf{1}_B\,\phi_1(W^{\delta,1})\cdots\phi_{k-1}(W^{\delta,k-1})\Big]\times\mathbb{N}_0^*(\phi_k\mid M>\delta),$$

and the proof of (3.47) is completed by an induction argument. □

3.7.2 Excursion debuts and discontinuities of the exit measure process

We start with a first proposition that relates levels of excursion debuts to discontinuity times for the process $(Z_x)_{x>0}$.

Proposition 3.7.3. $\mathbb{N}_0$ *a.e., discontinuity times for the process* $(Z_x)_{x>0}$ *are exactly all reals of the form* $-V_u$ *for* $u\in D$.

Proof. Recall that, for every $x\geq 0$ we have set

$$\mathcal{Y}_x=\int_0^\sigma \mathrm{d}s\,\mathbf{1}_{\{\tau_{-x}(W_s)=\infty\}}.$$

If (x_n) is a monotone increasing sequence that converges to $x>0$, then the indicator functions $\mathbf{1}_{\{\tau_{-x_n}(W_s)=\infty\}}$ converge to $\mathbf{1}_{\{\tau_{-x}(W_s)=\infty\}}$, and by dominated convergence it follows that $(\mathcal{Y}_x)_{x>0}$ has left-continuous sample paths. On the other hand, if (x_n) is a monotone decreasing sequence that converges to $x>0$, with $x_n>x$ for every n, one immediately gets that

$$\int_0^\sigma \mathrm{d}s\,\mathbf{1}_{\{\tau_{-x_n}(W_s)=\infty\}}\underset{n\to\infty}{\longrightarrow}\int_0^\sigma \mathrm{d}s\,\mathbf{1}_{\{\underline{W}_s\geq -x\}}.$$

It follows that $(\mathcal{Y}_x)_{x>0}$ also has right limits, and that x is a discontinuity point of $\mathcal{Y}$ if and only if

$$\int_0^\sigma \mathrm{d}s\,\mathbf{1}_{\{\underline{W}_s=-x\}}>0.$$

The latter condition holds if and only if there exists $s\in[0,\sigma]$ such that $\hat W_s>-x$ and $\underline{W}_s=-x$ (we use the fact that $\mathbb{N}_0$ a.e. for every $y\in\mathbb{R}$, $\int_0^\sigma \mathrm{d}s\,\mathbf{1}_{\{\hat W_s=y\}}=0$, which follows from the existence of local times for the tip process of the Brownian snake, see e.g. [15]). However, the existence of $s\in[0,\sigma]$ such that $\hat W_s>-x$ and $\underline{W}_s=-x$ implies that there is an excursion debut u with $V_u=-x$, and the converse is also true. Summarizing, we have obtained that discontinuity times for the process $(\mathcal{Y}_x)_{x>0}$ are exactly all reals of the form $-V_u$ for $u\in D$.

To complete the proof of the proposition, we use the fact that discontinuity times for $(\mathcal{Y}_x)_{x>0}$ are the same as discontinuity times for $(Z_x)_{x>0}$, as a consequence of Corollary 4.9 in [25] which essentially identifies the joint distribution of this pair of processes. To be precise the latter result is not concerned with the processes Z and $\mathcal{Y}$ under $\mathbb{N}_0$ but with superpositions of these processes corresponding to a Poisson measure with intensity $\mathbb{N}_0$. A simple argument however shows that this implies the result we need. □

We now identify the value of the jump of the process Z at the time $-V_u$ when $u \in D$. Note that Proposition 3.7.1 allows us to make sense of the exit measure $Z_0^*(W^{(u)})$ for any $u \in D$.

Proposition 3.7.4. $\mathbb{N}_0$ *a.e. for every* $u \in D$*, the jump of the process* Z *at time* $-V_u$ *is equal to* $Z_0^*(W^{(u)})$.

Proof. We fix $\delta > 0$, and we will prove that the assertion of the proposition holds $\mathbb{N}_0^{(\beta)}$ a.e. when $u = u_1^\delta$, the first excursion debut with level smaller than $-\beta$ and height greater than δ, on the event $\{N_\delta \geq 1\}$. Since $\beta > 0$ is arbirary it follows that the result holds for the first excursion debut with height greater than δ (if it exists). Using the special Markov property one then gets that it also holds for the second excursion debut with height greater than δ, and by induction for all excursion debuts with height greater than δ. Since δ is arbitrary, this gives the desired result for every $u \in D$.

So from now on we focus on the case $u = u_1^\delta$, and in what follows we restrict our attention to the event $\{N_\delta \geq 1\}$, so that u_1^δ is well defined. Recall that for integers $n \geq 1$ and $k \geq 1$, $(\omega_i^{k,2^{-n}})_{i \in I_k^{2^{-n}}}$ is the collection of excursions of the Brownian snake outside $(-k2^{-n}, \infty)$, and we use the notation $\tilde{\omega}_i^{k,2^{-n}}$ for $\omega_i^{k,2^{-n}}$ truncated at level $-(k+1)2^{-n}$ (and translated so that its starting point is 2^{-n}). Let n_0 be the first integer such that $2^{n_0}\beta \geq 1$. From now on we consider values of n such that $n \geq n_0$. We define $H_n = \lfloor -2^n V_{u_1^\delta} \rfloor \geq 1$, in such a way that

$$H_n \, 2^{-n} \leq -V_{u_1^\delta} < (H_n + 1)2^{-n}. \tag{3.48}$$

If we set for $\omega \in \mathcal{S}$,

$$O(\omega) = \sup\{\hat{W}_s - \underline{W}_s : 0 \leq s \leq \sigma\},$$

then H_n is the first integer $k \geq 1$ such that $O(\tilde{\omega}_i^{k,2^{-n}}) > \delta$ for some $i \in I_k^{2^{-n}}$. This index i may be not unique, and for this reason we introduce the event $A_n \subset \{N_\delta \geq 1\}$ where the property $O(\tilde{\omega}_i^{k,2^{-n}}) > \delta$ holds for exactly one index $i = i_n \in I_{H_n}^{2^{-n}}$. On the event A_n, we let $\omega_{(n)} = \tilde{\omega}_{i_n}^{H_n,2^{-n}}$ be the corresponding excursion and on the complement of A_n we let $\omega_{(n)}$ be the trivial snake path with duration 0 in $\mathcal{S}_0$. Notice that, on the event A_n, the excursion debut u_1^δ must then belong to (the subtree coded by the interval corresponding to) the excursion $\omega_{i_n}^{H_n,2^{-n}}$. We also note that the sequence $(A_n)_{n \geq n_0}$ is monotone increasing, and that $\mathbb{N}_0^{(\beta)}(A_n \mid N_\delta \geq 1)$ converges to 1 as $n \to \infty$ because there cannot be two excursion debuts at the same level. Furthermore, by the special Markov property, the distribution of $\omega_{(n)}$ under $\mathbb{N}_0^{(\beta)}(\cdot \mid A_n)$ is the law of $\tilde{W}$ under $\mathbb{N}_{2^{-n}}(\cdot \mid O(\tilde{W}) > \delta)$.

We then note that, for every $n \geq n_0$, we have on the event A_n,

$$Z_{(H_n+1)2^{-n}} = \sum_{i \in I_{H_n}^{2^{-n}}} \mathcal{Z}_0(\tilde{\omega}_i^{H_n,2^{-n}}) = \mathcal{Z}_0(\omega_{(n)}) + \sum_{i \in I_{H_n}^{2^{-n}}, i \neq i_n} \mathcal{Z}_0(\tilde{\omega}_i^{H_n,2^{-n}}). \tag{3.49}$$

To simplify notation, we write $b = -V_{u_1^\delta}$. We claim that

$$\sum_{i \in I_{H_n}^{2^{-n}}, i \neq i_n} \mathcal{Z}_0(\tilde{\omega}_i^{H_n,2^{-n}}) \underset{n \to \infty}{\longrightarrow} Z_{b-}, \tag{3.50}$$

where the convergence holds in probability under $\mathbb{N}_0^{(\beta)}(\cdot \mid N_\delta \geq 1)$ – the fact that i_n is only defined on A_n creates no problem here since $\mathbb{N}_0^{(\beta)}(A_n \mid N_\delta \geq 1)$ converges to 1.

Proof of (3.50). It will be convenient to introduce the point measure

$$\tilde{\mathcal{N}}_k^{2^{-n}} = \sum_{i \in I_k^{2^{-n}}} \delta_{\tilde{\omega}_i^{k,2^{-n}}},$$

for every $n \geq 1$ and $k \geq 1$. We first observe that, on the event A_n, we have the equality

$$\sum_{i \in I_{H_n}^{2^{-n}}, i \neq i_n} \mathcal{Z}_0(\tilde{\omega}_i^{H_n,2^{-n}}) = \int_{\{O \leq \delta\}} \tilde{\mathcal{N}}_{H_n}^{2^{-n}}(\mathrm{d}\omega)\, \mathcal{Z}_0(\omega).$$

Since $\mathbb{N}_0^{(\beta)}(A_n \mid N_\delta \geq 1)$ converges to 1, the proof of (3.50) reduces to checking that

$$\int_{\{O \leq \delta\}} \tilde{\mathcal{N}}_{H_n}^{2^{-n}}(\mathrm{d}\omega)\, \mathcal{Z}_0(\omega) \underset{n\to\infty}{\longrightarrow} Z_{b-}.$$

Since $2^{-n}H_n \uparrow b$, we have $Z_{2^{-n}H_n} \longrightarrow Z_{b-}$, a.e. under $\mathbb{N}_0^{(\beta)}(\cdot \mid N_\delta \geq 1)$, and so it is enough to prove that

$$\int_{\{O \leq \delta\}} \tilde{\mathcal{N}}_{H_n}^{2^{-n}}(\mathrm{d}\omega)\, \mathcal{Z}_0(\omega) - Z_{2^{-n}H_n} \underset{n\to\infty}{\longrightarrow} 0.$$

Note that we may have $H_n = \lfloor -2^n V_{u_1^\delta} \rfloor < 2^n\beta$ although $-V_{u_1^\delta} \geq \beta$, but this occurs with $\mathbb{N}_0^{(\beta)}$-probability tending to 0. Thanks to this observation, the preceding convergence will hold provided that, for every $\varepsilon > 0$, the quantities in the next display tend to 0 as $n \to \infty$:

$$\begin{aligned}
&\mathbb{N}_0^{(\beta)}\left(\left\{ \left| \int_{\{O \leq \delta\}} \tilde{\mathcal{N}}_{H_n}^{2^{-n}}(\mathrm{d}\omega)\, \mathcal{Z}_0(\omega) - Z_{2^{-n}H_n} \right| > \varepsilon \right\} \cap \{N_\delta \geq 1\} \cap \{H_n \geq 2^n\beta\} \right) \\
&\quad = \sum_{k \geq 2^n\beta} \mathbb{N}_0^{(\beta)}\left(\left\{ \left| \int_{\{O \leq \delta\}} \tilde{\mathcal{N}}_k^{2^{-n}}(\mathrm{d}\omega)\, \mathcal{Z}_0(\omega) - Z_{k2^{-n}} \right| > \varepsilon \right\} \cap \{N_\delta \geq 1\} \cap \{H_n = k\} \right) \\
&\quad = \sum_{k \geq 2^n\beta} \mathbb{N}_0^{(\beta)}\left(\left\{ \left| \int_{\{O \leq \delta\}} \tilde{\mathcal{N}}_k^{2^{-n}}(\mathrm{d}\omega)\, \mathcal{Z}_0(\omega) - Z_{k2^{-n}} \right| > \varepsilon \right\} \cap \{\tilde{\mathcal{N}}_k^{2^{-n}}(O > \delta) \geq 1\} \cap \{H_n \geq k\} \right).
\end{aligned}$$

The last equality holds because the event $\{N_\delta \geq 1\} \cap \{H_n = k\}$ coincides with $\{\tilde{\mathcal{N}}_k^{2^{-n}}(O > \delta) \geq 1\} \cap \{H_n \geq k\}$. Next we notice that the event $\{H_n \geq k\}$ is $\mathcal{E}^{(-k2^{-n},\infty)}$-measurable and that, under $\mathbb{N}_0^{(\beta)}$, conditionally on $\mathcal{E}^{(-k2^{-n},\infty)}$, $\tilde{\mathcal{N}}_k^{2^{-n}}$ is a Poisson measure whose intensity is $Z_{k2^{-n}}$ times the "law" of $\tilde{W}$ under $\mathbb{N}_{2^{-n}}$. It follows that the quantities in the last display are also equal to

$$\sum_{k \geq 2^n\beta} \mathbb{N}_0^{(\beta)}\Big(\psi_\varepsilon^n(Z_{k2^{-n}})\, \mathbf{1}_{\{\tilde{\mathcal{N}}_k^{2^{-n}}(O>\delta)\geq 1\} \cap \{H_n \geq k\}} \Big), \tag{3.51}$$

where, for every $a \geq 0$,

$$\psi_\varepsilon^n(a) = P\left(\left| \int_{\{O \leq \delta\}} \mathcal{N}_{n,a}(\mathrm{d}\omega)\, \mathcal{Z}_0(\omega) - a \right| > \varepsilon \right),$$

if $\mathcal{N}_{n,a}$ denotes a Poisson measure whose intensity is a times the "law" of $\tilde{W}$ under $\mathbb{N}_{2^{-n}}$. It is easy to verify that $\psi_\varepsilon^n(a)$ tends to 0 as $n \to \infty$, for every fixed a. First note that we can remove the restriction to $\{O \leq \delta\}$ since $P(\mathcal{N}_{n,a}(O > \delta) > 0)$ tends to 0. Then we just have to observe that $\int \mathcal{N}_{n,a}(\mathrm{d}\omega)\, \mathcal{Z}_0(\omega)$ converges in probability to a as $n \to \infty$, as a straightforward consequence of (3.9). Furthermore, a simple monotonicity argument shows that the convergence of $\psi^n_\varepsilon(a)$ to 0 holds uniformly on every compact subset of $\mathbb{R}_+$.

Finally, using again the fact that $\{\tilde{\mathcal{N}}^{2^{-n}}_k(O > \delta) \geq 1\} \cap \{H_n \geq k\} = \{N_\delta \geq 1\} \cap \{H_n = k\}$, the quantity in (3.51) is bounded by

$$\mathbb{N}^{(\beta)}_0\Big(\psi^n_\varepsilon(Z_{2^{-n}H_n})\,\mathbf{1}_{\{N_\delta \geq 1\}}\Big),$$

and this tends to 0 as $n \to \infty$ by the previous observations. This completes the proof of our claim (3.50).

Let us complete the proof of the proposition. We already noticed that the distribution of $\omega_{(n)}$ under $\mathbb{N}^{(\beta)}_0(\cdot \mid A_n)$ is the law of $\tilde{W}$ under $\mathbb{N}_{2^{-n}}(\cdot \mid O(\tilde{W}) > \delta)$. We observe that, for every $\varepsilon > 0$, the following inclusions hold $\mathbb{N}_\varepsilon$ a.e.

$$\{\tilde{M} > \delta + \varepsilon\} \subset \{O(\tilde{W}) > \delta\} \subset \{\tilde{M} > \delta\}$$

and moreover the ratio $\mathbb{N}_\varepsilon(\tilde{M} \geq \delta + \varepsilon)/\mathbb{N}_\varepsilon(\tilde{M} > \delta)$ tends to 1 as $\varepsilon \to 0$. It follows that the result of Proposition 3.6.3 remains valid if the conditioning by $\{\tilde{M} > \delta\}$ is replaced by $\{O(\tilde{W}) > \delta\}$. Thanks to this simple observation, we can deduce from Proposition 3.6.3 that

$$(\omega_{(n)}, \mathcal{Z}_0(\omega_{(n)})) \xrightarrow[n\to\infty]{\text{(d)}} (W^{\delta,0}, Z^*_0(W^{\delta,0})), \tag{3.52}$$

where $W^{\delta,0}$ is distributed according to $\mathbb{N}^*_0(\cdot \mid M > \delta)$ and the convergence holds in distribution under $\mathbb{N}^{(\beta)}_0(\cdot \mid N_\delta \geq 1)$. Furthermore, from the last assertion of Lemma 3.3.8, and the fact that $\omega^{H_n,2^{-n}}_{i_n}$ is the excursion outside $(-H_n 2^{-n}, \infty)$ that "contains" u^δ_1, we get that $\omega_{(n)}$ converges $\mathbb{N}^{(\beta)}_0$ to $W^{(u^\delta_1)}$, a.e. on $\{N_\delta \geq 1\}$. On the other hand, (3.48) and the right-continuity of sample paths of Z imply that

$$Z_{(H_n+1)2^{-n}} \xrightarrow[n\to\infty]{} Z_b, \tag{3.53}$$

$\mathbb{N}^{(\beta)}_0$ a.s. on $\{N_\delta \geq 1\}$. Then, using (3.49), (3.50) and (3.53), we immediately get that $\mathcal{Z}_0(\omega_{(n)})$ converges to the random variable $Z_b - Z_{b-}$, in probability under $\mathbb{N}^{(\beta)}_0(\cdot \mid N_\delta \geq 1)$. So we know that the pair $(\omega_{(n)}, \mathcal{Z}_0(\omega_{(n)}))$ converges in probability to $(W^{(u^\delta_1)}, Z_b - Z_{b-})$ under $\mathbb{N}^{(\beta)}_0(\cdot \mid N_\delta \geq 1)$, and it follows from (3.52) that the law of $(W^{(u^\delta_1)}, Z_b - Z_{b-})$ under $\mathbb{N}^{(\beta)}_0(\cdot \mid N_\delta \geq 1)$ is the law of $(W^{\delta,0}, Z^*_0(W^{\delta,0}))$. This forces $Z_b - Z_{b-} = Z^*_0(W^{(u^\delta_1)}))$, which completes the proof. $\square$

3.7.3 The Poisson process of excursions

The following proposition is reminiscent of Itô's famous Poisson point process of excursions of linear Brownian motion. We recall that $\beta > 0$ is fixed and that $u^\delta_1, \ldots, u^\delta_{N_\delta}$ are the successive excursion debuts with height greater than δ and level smaller than $-\beta$.

Proposition 3.7.5. *We can find an auxiliary probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *such that, on the product space* $\Omega \times \mathcal{S}$ *equipped with the probability measure* $\mathbb{P} \otimes \mathbb{N}^{(\beta)}_0$*, we can construct a Poisson measure* $\mathcal{P}$ *on*

$\mathbb{R}_+ \times \mathcal{S}$ *with intensity* $\mathrm{d}t \otimes \mathbb{N}_0^*(\mathrm{d}\omega)$ *so that the following holds. For every* $\delta > 0$, *if* $(t_1^\delta, \omega_1^\delta), (t_2^\delta, \omega_2^\delta), \ldots$ *is the sequence of atoms of the measure* $\mathcal{P}(\cdot \cap (\mathbb{R}_+ \times \{M > \delta\})$, *ranked so that* $t_1^\delta < t_2^\delta < \cdots$, *we have* $\tilde{W}^{(u_i^\delta)} = \omega_i^\delta$ *for every* $1 \leq i \leq N_\delta$. *Furthermore, the Poisson measure* $\mathcal{P}$ *is independent of* $\mathcal{E}^{(-\beta,\infty)}$.

This proposition means that all excursions above the minimum (with level smaller than β) can be viewed as the atoms of a certain Poisson point process. In contrast with the classical Itô theorem of excursion theory for Brownian motion, we need to enlarge the underlying probability space in order to construct the Poisson measure $\mathcal{P}$.

Proof. We first explain how we can choose the auxiliary random variables $\overline{W}^{\delta,j}$ of Lemma 3.7.2 in a consistent way when δ varies. We set $\delta_k = 2^{-k}$ for every $k \geq 1$ and we restrict our attention to values of δ in the sequence $(\delta_k)_{k\geq 1}$. On an auxiliary probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\overline{\mathcal{P}}$ be a Poisson measure on $\mathbb{R}_+ \times \mathcal{S}$ with intensity $\mathrm{d}t \otimes \mathbb{N}_0^*(\mathrm{d}\omega)$. For every $k \geq 1$, let $(\bar{t}^{k,j}, \overline{W}^{k,j})_{j\geq 1}$ be the sequence of atoms of $\overline{\mathcal{P}}$ that fall in the set $\mathbb{R}_+ \times \{M > \delta_k\}$ (ordered so that $\bar{t}^{k,1} < \bar{t}^{k,2} < \cdots$). Then, for every $k \geq 1$, $(\overline{W}^{k,1}, \overline{W}^{k,2}, \ldots)$ forms an i.i.d. sequence of variables distributed according to $\mathbb{N}_0^*(\cdot \mid M > \delta_k)$. By Lemma 3.7.2, under the product probability measure $\mathbb{P} \otimes \mathbb{N}_0^{(\beta)}$, the sequence $(W^{k,1}, W^{k,2}, \ldots)$ defined by

$$W^{k,j} = \begin{cases} \tilde{W}^{(u_j^{\delta_k})} & \text{if } 1 \leq j \leq N_\delta \\ \overline{W}^{k,j-M_\delta} & \text{if } j > N_\delta \end{cases}$$

is also a sequence of i.i.d. random variables distributed according to $\mathbb{N}_0^*(\cdot \mid M > \delta_k)$, and is independent of the σ-field $\mathcal{E}^{(-\beta,\infty)}$.

Obviously, if $k < k'$, the excursions $\tilde{W}^{(u_j^{\delta_k})}$ $1 \leq j \leq N_{\delta_k}$ are obtained by considering the elements of the finite sequence $\tilde{W}^{(u_j^{\delta_{k'}})}$, $1 \leq j \leq N_{\delta_{k'}}$ that belong to the set $\{M > \delta_k\}$, and similarly the sequence $(\overline{W}^{k,j})_{j\geq 1}$ consists of those terms of the sequence $(\overline{W}^{k',j})_{j\geq 1}$ that belong to the set $\{M > \delta_k\}$. It follows that, for every $k < k'$, the sequence $(W^{k,j})_{j\geq 1}$ is obtained by keeping only those terms of the sequence $(W^{k',j})_{j\geq 1}$ that belong to the set $\{M > \delta_k\}$. Note that the law of the collection

$$(W^{k,j})_{j\geq 1, k\geq 1}$$

is then completely determined by this consistency property and the fact that, for every fixed $k \geq 1$, $(W^{k,j})_{j\geq 1}$ is a sequence of i.i.d. random variables distributed according to $\mathbb{N}_0^*(\cdot \mid M > \delta_k)$. In particular,

$$(W^{k,j})_{j\geq 1, k\geq 1} \overset{(\mathrm{d})}{=} (\overline{W}^{k,j})_{j\geq 1, k\geq 1} \tag{3.54}$$

Also note that the collection $(W^{k,j})_{j\geq 1, k\geq 1}$ is independent of the σ-field $\mathcal{E}^{(-\beta,\infty)}$.

It is a simple exercise on Poisson measures to verify that $\overline{\mathcal{P}}$ is equal a.s. to a measurable function of the collection $(\overline{W}^{k,j})_{j\geq 1, k\geq 1}$. Indeed, it suffices to verify that the times $(\bar{t}^{k,j})_{j\geq 1, k\geq 1}$ are (a.s.) measurable functions of this collection. Let us outline the argument in the case $k = j = 1$. If, for every $k \geq 1$, we write

$$m_k := \#\{j \geq 1 : \bar{t}^{k,j} < \bar{t}^{1,1}\}$$

then m_k is just the number of terms in the sequence $(\overline{W}^{k,j})_{j\geq 1}$ before the first term that belongs to $\{M > \delta_1\}$, and is thus a function of $(\overline{W}^{k,j})_{j\geq 1,k\geq 1}$. Elementary arguments using Lemma 3.3.9 show that we have the almost sure convergence

$$\mathbb{N}_0^*(M > \delta_k)^{-1}\, m_k \underset{k\to\infty}{\longrightarrow} \bar{t}^{1,1},$$

thus giving the desired measurability property.

So there exists a measurable function Φ such that we have a.s.

$$\overline{\mathcal{P}} = \Phi\Big((\overline{W}^{k,j})_{j\geq 1,k\geq 1}\Big).$$

Then we can just set

$$\mathcal{P} = \Phi\Big((W^{k,j})_{j\geq 1,k\geq 1}\Big).$$

By (3.54), $\mathcal{P}$ has the same distribution as $\overline{\mathcal{P}}$. By construction, the properties stated in the proposition hold when $\delta = \delta_k$, for every $k \geq 1$. This implies that they hold for every $\delta > 0$. □

In what follows, we will use not only the statement of Proposition 3.7.5 but also the explicit construction of $\mathcal{P}$ that is given in the preceding proof (we did not include this explicit construction in the statement of Proposition 3.7.5 for the sake of conciseness).

We now state an important lemma, which shows that the process $(Z_{\beta+r})_{r\geq 0}$ can be recovered from (Z_β and) the Poisson measure $\mathcal{P}$. To this end, we introduce the point measure $\mathcal{P}^\circ$ defined as the image of $\mathcal{P}$ under the mapping $(t,\omega) \longrightarrow (t, Z_0^*(\omega))$. From the form of the "law" of Z_0^* under $\mathbb{N}_0^*$ given in Proposition 3.6.2, $\mathcal{P}^\circ$ is (under $\mathbb{P}\otimes\mathbb{N}_0^{(\beta)}$) a Poisson measure on $\mathbb{R}_+ \times (0,\infty)$ with intensity

$$\mathrm{d}t \otimes \sqrt{\frac{3}{2\pi}} z^{-5/2}\,\mathrm{d}z.$$

We can associate with this point measure a centered Lévy process $U = (U_t)_{t\geq 0}$ (with no negative jump) started from 0, such that

$$\sum_{t\in\mathcal{D}_U} \delta_{(t,\Delta U_t)} = \mathcal{P}^\circ,$$

where $\mathcal{D}_U$ is the set of discontinuity times of U. Note that the Laplace transform of U_t is

$$E[\exp(-\lambda U_t)] = \exp(t\psi(\lambda)),$$

where

$$\psi(\lambda) = \sqrt{\frac{3}{2\pi}} \int_0^\infty (e^{-\lambda z} - 1 + \lambda z)\, z^{-5/2}\,\mathrm{d}z = \sqrt{\frac{8}{3}}\,\lambda^{3/2}.$$

Notice that we get the same function $\psi(\lambda)$ as in subsection 3.2.5.

Lemma 3.7.6. *Set $X_t = Z_\beta + U_t$ for every $t \geq 0$. Then, we have, $\mathbb{P}\otimes\mathbb{N}_0^{(\beta)}$ a.s.,*

$$Z_{\beta+r} = X_{\inf\{t\geq 0:\int_0^t (X_s)^{-1}\mathrm{d}s > r\}}\,, \quad \textit{for every } 0 \leq r < -W_* - \beta.$$

Remark. We have $Z_r = 0$ for every $r \geq -W_*$, so that the formula of the lemma indeed expresses $(Z_{\beta+r})_{r\geq 0}$ as a function of X, which is itself defined in terms of Z_β and the point measure $\mathcal{P}^\circ$.

Proof. First notice that $(U_t)_{t\geq 0}$ is independent of Z_β because $\mathcal{P}$ is independent of $\mathcal{E}^{(-\beta,\infty)}$. Therefore, $(X_t)_{t\geq 0}$ is a Lévy process started from Z_β. On the other hand, we know that $(Z_{\beta+r})_{r\geq 0}$ is under $\mathbb{N}_0^{(\beta)}$ a continuous-state branching process with branching mechanism ψ. By the classical Lamperti transformation (see e.g. [13]), if we set $T_0' := \int_0^\infty Z_{\beta+t}\,\mathrm{d}t$ and, for every $0 \leq r < T_0'$,

$$X_r' := Z_{\beta+\inf\{s\geq 0:\int_0^s Z_{\beta+t}\,\mathrm{d}t > r\}}, \tag{3.55}$$

the process $(X_r')_{0\leq r<T_0'}$ has the same distribution as $(X_r)_{0\leq r<T_0}$, where $T_0 := \inf\{t\geq 0 : X_t = 0\}$. Furthermore, by inverting (3.55), we have also

$$Z_{\beta+r} = X'_{\inf\{t\geq 0:\int_0^t (X_s')^{-1}\mathrm{d}s > r\}}, \quad \text{for every } 0\leq r < T_0^Z, \tag{3.56}$$

where $T_0^Z = -W_* - \beta$ is the hitting time of 0 by Z.

Comparing (3.56) with the statement of the lemma, we see that we only need to verify that we have the a.s. equality $(X_r)_{0\leq r<T_0} = (X_r')_{0\leq r<T_0'}$. To this end, recall the Poisson measure $\overline{\mathcal{P}}$ in the proof of Proposition 3.7.5. We define $\overline{\mathcal{P}}^\circ$ as the image of $\overline{\mathcal{P}}$ under the mapping $(t,\omega)\longrightarrow (t, Z_0^*(\omega))$, and associate with $\overline{\mathcal{P}}^\circ$ a Lévy process $(\overline{U}_t)_{t\geq 0}$ having the same distribution as $(U_t)_{t\geq 0}$. We complete the definition of X' by setting for every $t\geq 0$,

$$X'_{T_0'+t} = \overline{U}_t.$$

We then observe that X and X' are two Lévy processes with the same distribution and the same (random) initial value Z_β. Furthermore, a.s. for every $\eta > 0$, the ordered sequence of jumps of size greater than η is the same for X' and for X. First note that the jumps of X' that occur before the hitting time of 0 are the same as the jumps of Z after time β, and, by Proposition 3.7.4, these are exactly the quantities $Z_0^*(\tilde{W}^{(u)})$ when u varies over the excursion debuts with level smaller than $-\beta$. Recalling our construction of X from the point measure $\mathcal{P}^\circ$, we obtain that, for every $\eta > 0$, the ordered sequence of jumps of X' of size greater than η that occur before the hitting time of 0 will also appear as the first n_η jumps of X of size greater than η, for some random integer n_η depending on η. Then, the ordered sequence of jumps of X' of size greater than η that occur after the hitting time of 0 consists of the quantities $Z_0^*(\omega)$ where (t,ω) varies over the atoms of $\overline{\mathcal{P}}$ such that $Z_0^*(\omega) > \eta$ and these quantities are ranked according to the values of t. Recalling the way $\mathcal{P}$ was defined, we see that the same sequence will appear as the sequence of jumps of X of size greater than η occurring after the n_η-th one.

Finally, once we know that, for every $\eta > 0$, the ordered sequence of jumps of size greater than η is the same for X' and for X, the fact that X and X' are two Lévy processes with the same distribution and the same initial value implies that they are a.s. equal, which completes the proof. □

3.7.4 The main theorem

Our main result identifies the conditional distribution of excursions above the minimum given the exit measure process Z. We let $\mathcal{D}_Z$ stand for the set of all jump times of Z. Recall from Proposition 3.7.3 that there is a one-to-one correspondence between $\mathcal{D}_Z$ and excursions above the minimum. If u is an excursion debut, and $r = -V_u$ is the associated element of $\mathcal{D}_Z$, we write $\tilde{W}^{(r)} = \tilde{W}^{(u)}$ in the following statement. We let $\mathbb{D}(0,\infty)$ stand for the usual Skorokhod space of càdlàg functions from $(0,\infty)$ into $\mathbb{R}$.

Theorem 3.7.7. *Let F be a nonnegative measurable function on $\mathbb{D}(0,\infty)$, and let G be a nonnegative measurable function on $\mathbb{R}_+ \times \mathcal{S}$. Then,*

$$\mathbb{N}_0\bigg(F(Z) \exp\Big(-\sum_{r\in\mathcal{D}_Z} G(r,\tilde{W}^{(r)})\Big)\bigg) = \mathbb{N}_0\bigg(F(Z) \prod_{r\in\mathcal{D}_Z} \mathbb{N}_0^*\Big(\exp(-G(r,\cdot)) \,\Big|\, Z_0^* = \Delta Z_r\Big)\bigg).$$

In other words, under $\mathbb{N}_0$ and conditionally on the exit measure process Z, the excursions above the minimum are independent, and, for every $r \in \mathcal{D}_Z$, the conditional law of the associated excursion is $\mathbb{N}_0^*(\cdot \mid Z_0^* = \Delta Z_r)$.

Proof. Let us a start with simple reductions of the proof. First we may assume that $\mathbb{N}_0(F(Z)) < \infty$ since the general case will follow by monotone convergence. Then, we may assume that $G(r,\omega) = 0$ if $r \le \gamma$, for some $\gamma > 0$, and it is also sufficient to prove that the statement holds when $\mathbb{N}_0$ is replaced by $\mathbb{N}_0^{(\beta)}$ for some fixed $\beta > 0$. Finally, we may restrict the sum or the product over r to jump times such that $\Delta Z_r > \alpha$, for some fixed $\alpha > 0$.

In view of the preceding observations, we only need to verify that, for every $\alpha > 0$ and $\beta > 0$,

$$\mathbb{N}_0^{(\beta)}\Bigg(F(Z) \exp\Bigg(-\sum_{\substack{r\in\mathcal{D}_Z^{(\beta)}\\ \Delta Z_r>\alpha}} G(r,\tilde{W}^{(r)})\Bigg)\Bigg) = \mathbb{N}_0^{(\beta)}\Bigg(F(Z) \prod_{\substack{r\in\mathcal{D}_Z^{(\beta)}\\ \Delta Z_r>\alpha}} \mathbb{N}_0^*\Big(\exp(-G(r,\cdot)) \,\Big|\, Z_0^* = \Delta Z_r\Big)\Bigg),$$

where $\mathcal{D}_Z^{(\beta)} = \mathcal{D}_Z \cap (\beta,\infty)$.

From now on, we fix $\alpha > 0$ and $\beta > 0$. We will use the notation and definitions of the previous subsections, where $\beta > 0$ was fixed and we argued under $\mathbb{N}_0^{(\beta)}$. In particular it will be convenient to consider the product probability measure $\mathbb{P} \otimes \mathbb{N}_0^{(\beta)}$ as in subsection 3.7.3. Recall also the definition of the Poisson measure $\mathcal{P}$ and of the process X in Lemma 3.7.6 (these objects depend on the choice of β, which is fixed here), and the notation $T_0 = \inf\{t \ge 0 : X_t = 0\}$.

The first step is to rewrite the quantity

$$\sum_{\substack{r\in\mathcal{D}_Z^{(\beta)}\\ \Delta Z_r>\alpha}} G(r,\tilde{W}^{(r)})$$

in a different form. Recall from Lemma 3.7.6 that every jump time r of Z after time β, hence every excursion debut u with level smaller than $-\beta$, corresponds to a jump time of X before time T_0, and is therefore associated with an atom (t,ω) of $\mathcal{P}$, with $t < T_0$, such that $\omega = \tilde{W}^{(u)}$ and $Z_0^*(\omega) = Z_0^*(\tilde{W}^{(u)}) = \Delta Z_r$ (Proposition 3.7.4). Then, let $(t_1^\alpha,\omega_1^\alpha), (t_2^\alpha,\omega_2^\alpha), \ldots$ be the ordered sequence of all atoms (t,ω) of $\mathcal{P}$ such that $Z_0^*(\omega) > \alpha$. Also set $n_\alpha = \max\{i \ge 1 : t_i^\alpha < T_0\}$. For every $1 \le i \le n_\alpha$, write $z_i^\alpha = Z_0^*(\omega_i^\alpha)$ and r_i^α for the jump time of Z corresponding to the jump z_i^α. We can rewrite

$$\sum_{\substack{r\in\mathcal{D}_Z^{(\beta)}\\ \Delta Z_r>\alpha}} G(r,\tilde{W}^{(r)}) = \sum_{i=1}^{n_\alpha} G(r_i^\alpha,\omega_i^\alpha).$$

Then, writing $E[\cdot]$ for the expectation under $\mathbb{P} \otimes \mathbb{N}_0^{(\beta)}$, we are led to evaluate

$$E\Big[F(Z) \exp\Big(-\sum_{i=1}^{n_\alpha} G(r_i^\alpha,\omega_i^\alpha)\Big)\Big].$$

We do this by conditioning first with respect to the σ-field $\mathcal{H}$ generated by $\mathcal{E}^{(-\beta,\infty)}$ and the point measure $\mathcal{P}^\circ$. Notice that the process Z is measurable with respect to $\mathcal{H}$ (because U is obviously a measurable function of $\mathcal{P}^\circ$, and we can use Lemma 3.7.6). The finite sequence $r_1^\alpha,\ldots,r_{n_\alpha}^\alpha$ is also measurable with respect to $\mathcal{H}$ as it is the sequence of jump times of Z (after time β) corresponding to jumps of size greater than α. In particular, n_α is measurable with respect to $\mathcal{H}$. Finally the quantities $z_1^\alpha,\ldots,z_{n_\alpha}^\alpha$ are the corresponding jumps and therefore also measurable with respect to $\mathcal{H}$.

On the other hand, by standard properties of Poisson measures, we know that the sequence $\omega_1^\alpha,\omega_2^\alpha,\ldots$ is a sequence of i.i.d. variables distributed according to $\mathbb{N}_0^*(\cdot\mid Z_0^*>\alpha)$. Recalling that $\mathcal{P}$ is independent of $\mathcal{E}^{(-\beta,\infty)}$, we see that conditioning on the σ-field $\mathcal{H}$ has the effect of conditioning on the values of $Z_0^*(\omega_1^\alpha), Z_0^*(\omega_2^\alpha),\ldots$. Put in a more precise way, the conditional distribution of $\omega_1^\alpha,\omega_2^\alpha,\ldots$ knowing $\mathcal{H}$ is the distribution of a sequence of independent variables distributed respectively according to $\mathbb{N}_0^*(\cdot\mid Z_0^*=z_1^\alpha)$, $\mathbb{N}_0^*(\cdot\mid Z_0^*=z_2^\alpha),\ldots$.

By combining the preceding considerations, we get

$$E\Big[F(Z)\,\exp\Big(-\sum_{i=1}^{n_\alpha}G(r_i^\alpha,\omega_i^\alpha)\Big)\Big]=E\Big[F(Z)\prod_{i=1}^{n_\alpha}\mathbb{N}_0^*\big(\exp(-G(r_i^\alpha,\cdot)\mid Z_0^*=z_i^\alpha\big)\Big].$$

Now note that, with our definitions,

$$\prod_{i=1}^{n_\alpha}\mathbb{N}_0^*\big(\exp(-G(r_i^\alpha,\cdot)\mid Z_0^*=z_i^\alpha\big)=\prod_{\substack{r\in\mathcal{D}_Z^{(\beta)}\\ \Delta Z_r>\alpha}}\mathbb{N}_0^*\Big(\exp(-G(r,\cdot))\,\Big|\,Z_0^*=\Delta Z_r\Big),$$

so that the proof of the theorem is complete. □

3.8 Excursions away from a point

In this section, we briefly explain how we can derive the results stated in the introduction from our statements concerning excursions above the minimum. This relies on the famous theorem of Lévy stating that, if $(B_t)_{t\geq 0}$ is a linear Brownian motion starting from 0, and if $(L_t^0(B))_{t\geq0}$ is its local time process at 0, then the pair of processes

$$(B_t-\min\{B_r:0\leq r\leq t\},-\min\{B_r:0\leq r\leq t\})_{t\geq0}$$

has the same distribution as $(|B_t|,L_t^0(B))_{t\geq0}$. Notice that $L_t^0(B)$ can also be interpreted as the local time of $|B|$ at 0, provided we consider here the "symmetric local time", namely

$$L_t^0(|B|)=\lim_{\varepsilon\to0}\frac{1}{2\varepsilon}\int_0^t\mathbf{1}_{[-\varepsilon,\varepsilon]}(|B_r|)\,\mathrm{d}r.$$

Lévy's identity will show that (absolute values of) excursions away from 0 for our tree-indexed process have the same distribution as excursions above the minimum, which is essentially what we need to derive the results stated in the introduction.

Let us explain this in greater detail. For any finite path $\mathrm{w}\in\mathcal{W}_0$, define two other finite paths $\mathrm{w}^\bullet$ and $\ell_{\mathrm{w}}^\bullet$ with the same lifetime as w by the formulas

$$\begin{aligned}\mathrm{w}^\bullet(t)&:=\mathrm{w}(t)-\min\{\mathrm{w}(r):0\leq r\leq t\}\\ \ell_{\mathrm{w}}^\bullet(t)&:=-\min\{\mathrm{w}(r):0\leq r\leq t\}.\end{aligned}$$

On our canonical space $\mathcal{S}_0$ of snake trajectories, we can then make sense of $W_s^\bullet$ and $\ell^\bullet_{W_s}$ for every $s \geq 0$, and we write $L_s^\bullet = \ell^\bullet_{W_s}$ to simplify notation. Then, under $\mathbb{N}_0$, the pair $(W_s^\bullet, L_s^\bullet)_{s\geq 0}$ defines a random element of the space of two-dimensional snake trajectories with initial point $(0,0)$ (the latter space is defined by an obvious extension of Definition 3.2.2). Thanks to Lévy's theorem recalled above, it is then a simple matter to verify that the "law" of the pair $(W_s^\bullet, L_s^\bullet)_{s\geq 0}$ under $\mathbb{N}_0$ is the excursion measure from the point $(0,0)$ of the Brownian snake whose spatial motion is the Markov process $(|B_t|, L_t^0(B))$. We refer to [44, Chapter 4] for the definition of the Brownian snake associated with a general spatial motion and of its excursion measures. In a way similar to the beginning of Section 3.3, we then set

$$V_u^\bullet = \hat{W}_s^\bullet = \hat{W}_s - \min\{W_s(t) : 0 \leq t \leq \zeta_s\} = V_u - \min\{V_v : v \in [\![\rho, u]\!]\},$$

for every $u \in \mathcal{T}_\zeta$ and $s \geq 0$ such that $p_\zeta(s) = u$.

Say that $u \in \mathcal{T}_\zeta$ is an excursion debut away from 0 for $V^\bullet$ if

(i) $V_u^\bullet = 0$;

(ii) u has a strict descendant w such that $V_v^\bullet \neq 0$ for all $v \in]\!]\rho, w]\!]$.

Then it follows from our definitions that u is an excursion debut away from 0 for $V^\bullet$ if and only if u is an excursion debut above the minimum in the sense of Section 3.3, that is, if and only if $u \in D$. Furthermore, Proposition 3.3.4 shows that the connected components of the open set $\{u \in \mathcal{T}_\zeta : V_u^\bullet > 0\}$ are exactly the sets $\mathrm{Int}(C_u)$, $u \in D$. Furthermore, for every $u \in D$, the values of $V^\bullet$ over C_u are described by the snake trajectory $\tilde{W}^{(u)}$ (which can thus be viewed as the excursion of $V^\bullet$ away from 0 corresponding to u).

In order to recover the setting of the introduction, we still need to assign signs to the excursions of $V^\bullet$ away from 0. To this end, we let $(v_1, v_2, \ldots)$ be a measurable enumeration of D – formally we should rather enumerate times $s_1, s_2, \ldots$ such that $p_\zeta(s_1) = v_1, p_\zeta(s_2) = v_2, \ldots$. On an auxiliary probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we then consider a sequence $(\xi_1, \xi_2, \ldots)$ of i.i.d. random variables such that

$$\mathbb{P}(\xi_i = 1) = \mathbb{P}(\xi_i = -1) = \frac{1}{2}$$

for every $i \geq 1$. Under the product measure $\mathbb{P} \otimes \mathbb{N}_0$, we then set, for every $u \in \mathcal{T}_\zeta$,

$$V_u^* := \begin{cases} \xi_i V_u^\bullet & \text{if } u \in \mathrm{Int}(C_{v_i}) \text{ for some } i \geq 1, \\ 0 & \text{if } V_u^\bullet = 0. \end{cases}$$

The fact that $u \mapsto V_u^\bullet$ is continuous implies that $u \mapsto V_u^*$ is also continuous on $\mathcal{T}_\zeta$. Furthermore the pair $(V^*_{p_\zeta(s)}, \zeta_s)$ is a tree-like path, and we denote the associated snake trajectory by $(W_s^*)_{s\geq 0}$. Then, the "law" of $(W_s^*)_{s\geq 0}$ under $\mathbb{P} \otimes \mathbb{N}_0$ is just the excursion measure $\mathbb{N}_0$. This is essentially a consequence of the fact that, starting from a process distributed as $(|B_t|)_{t\geq 0}$, one can reconstruct a linear Brownian motion started from 0 by assigning independently signs $+1$ or -1 with probability $1/2$ to the excursions away from 0. We omit the details.

Since the law of $(W_s^*)_{s\geq 0}$ under $\mathbb{P} \otimes \mathbb{N}_0$ is $\mathbb{N}_0$ we may replace the process $(W_s)_{s\geq 0}$ under $\mathbb{N}_0$ by the process $(W_s^*)_{s\geq 0}$ under $\mathbb{P} \otimes \mathbb{N}_0$ in order to prove the various statements of the introduction. To this end, we first notice that the excursion debuts away from 0 for V^* (obviously defined by properties (i) and (ii) with $V^\bullet$ replaced by V^*) are the same as the excursion debuts away from

0 for $V^\bullet$, and thus the same as excursion debuts above the minimum in the sense of Section 3.3. Moreover, for every $i = 1, 2, \ldots$, the excursion of V^* corresponding to v_i is described by

$$\tilde{W}^{*(v_i)} = \begin{cases} \tilde{W}^{(v_i)} & \text{if } \xi_i = 1, \\ -\tilde{W}^{(v_i)} & \text{if } \xi_i = -1. \end{cases}$$

In addition, if a_i is such that $p_\zeta(a_i) = v_i$, the local time at 0 of the path $W^*_{a_i}$ is equal to the (symmetric) local time at 0 of $|W^*_{a_i}| = W^\bullet_{a_i}$,

$$\ell^*_i = \hat{L}^\bullet_{a_i} = -\underline{W}_{a_i} = -V_{v_i}.$$

From the preceding remarks, it is now easy to derive Theorem 3.1.1 from Theorem 3.3.7. Indeed, the left hand side of the formula of Theorem 3.1.1 can be rewritten as

$$\mathbb{P} \otimes \mathbb{N}_0\Big(\sum_{i=1}^{\infty} \Phi(\ell^*_i, W^{*(v_i)}) \Big)$$

and, by the previous observations, the last display is equal to

$$\begin{aligned} \mathbb{P} \otimes \mathbb{N}_0\Big(\sum_{i=1}^{\infty} \Phi(-V_{v_i}, \xi_i \tilde{W}^{(v_i)}) \Big) &= \frac{1}{2}\, \mathbb{N}_0\Big(\sum_{i=1}^{\infty} (\Phi(-V_{v_i}, \tilde{W}^{(v_i)}) + \Phi(-V_{v_i}, -\tilde{W}^{(v_i)}) \Big) \\ &= \frac{1}{2} \int \mathbb{N}^*_0(\mathrm{d}\omega) \Big(\int_0^\infty \mathrm{d}x\, (\Phi(x,\omega) + \Phi(x, -\omega)) \Big) \end{aligned}$$

where the last equality is Theorem 3.3.7. This shows that Theorem 3.1.1 holds with $\mathbb{M}_0 = \frac{1}{2}(\mathbb{N}^*_0 + \check{\mathbb{N}}^*_0)$, where $\check{\mathbb{N}}^*_0$ is the image of $\mathbb{N}^*_0$ under $\omega \mapsto -\omega$. Then Proposition 3.1.2 follows from Proposition 3.6.1.

In order to derive Proposition 3.1.3, we note that, for every $r > 0$, the exit measure $\mathcal{X}_r$ of the snake $(W^\bullet, L^\bullet)$ outside the open set $\Delta_r := \mathbb{R}_+ \times [0, r)$ satisfies the following approximation $\mathbb{N}_0$ a.e.,

$$\mathcal{X}_r = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\zeta_s - \varepsilon < \tau_{\Delta_r}(W^\bullet_s, L^\bullet_s) < \zeta_s\}},$$

where $\tau_{\Delta_r}(W^\bullet_s, L^\bullet_s)$ stands for the first exit time from Δ_r of the path $(W^\bullet_s(t), L^\bullet_s(t))_{0 \le t \le \zeta_s}$. This is indeed the analog of the approximation result (3.8), and we already noticed that the latter approximation for the exit measure holds in a very general setting: see [44, Proposition V.1]. Coming back to the definition of $W^\bullet_s$ and $L^\bullet_s$ in terms of W_s, we see that we have

$$\mathcal{X}_r = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\zeta_s - \varepsilon < \tau_{-r}(W_s) < \zeta_s\}} = \mathcal{Z}_{-r},$$

where the last equality follows from (3.8). This simple remark allows us to identify the process $(\mathcal{X}_r)_{r>0}$ with the exit measure process $(Z_r)_{r>0}$, and justifies the observations preceding Proposition 3.1.3 in the introduction. Proposition 3.1.3 itself then follows from Propositions 3.7.3 and 3.7.4. Finally, Theorem 3.1.4 is a consequence of Theorem 3.7.7 and the fact that the excursions $\tilde{W}^{*(v_i)}$ can be written in the form $\xi_i\, \tilde{W}^{(v_i)}$, for $i = 1, 2, \ldots$.

Bibliographie

[1] Abraham, C., Rescaled bipartite planar maps converge to the Brownian map. *Ann. Inst. Henri Poincaré* (B), to appear.

[2] Abraham, C., Le Gall, J. F. Excursion theory for Brownian motion indexed by a Brownian tree, Preprint, availaible at `arXiv:1509.06616`

[3] Addario-Berry, L., Albenque, M., The scaling limit of random simple triangulations and random simple triangulations. Preprint, availaible at `arXiv:1306.5227`

[4] Aldous, D. The continuum random tree. I. *Ann. Probab.* **19(1)** (1991), 1–28.

[5] Aldous, D. The continuum random tree. II. An overview. In *Stochastic analysis (Durham, 1990)*, volume 167 of *London Math. Soc. Lecture Note Ser.*, Cambridge Univ. Press, Cambridge (1991), 23–70.

[6] Aldous, D. The continuum random tree III. *Ann. Probab.* **21(1)** (1993), 248–289.

[7] Aldous, D. Tree-based models for random distribution of mass. *J. Stat. Phys.* **73** (1993), 625–641.

[8] Angel, O., Schramm, O. Uniform infinite planar triangulation. *Comm. Math. Phys.* **241(2-3)** (2003), 191–213.

[9] Beltran, J., Le Gall, J.F., Quadrangulations with no pendant vertices. *Bernoulli* **19** (2013), 1150–1175.

[10] Bettinelli, J. Scaling Limit of Random Planar Quadrangulations with a Boundary. *Ann. Inst. Henri Poincaré* (B) **51** (2015), 432–477.

[11] Bettinelli, J., Jacob, E., Miermont, G. The scaling limit of uniform random plane maps, via the Ambjørn–Budd bijection. *Electronic J. Probab.* **19** (2014), 1–16.

[12] Bettinelli, J., Miermont, G. Compact Brownian surfaces I. Brownian disks. Preprint available at `arXiv:1507.08776`

[13] Bingham, N.H. Continuous branching processes and spectral positivity. *Stochastic Processes Appl.* **4** (1976), 217–242.

[14] Bingham, N.H., Goldie, C.M., Teugels, J.L., *Regular variation.* Cambridge University Press, 1989.

[15] Bousquet-Mélou, M., Janson, S., The density of the ISE and local limit laws for embedded trees. *Ann. Appl. Probab.* **16** (2006), 1597–1632.

[16] Bouttier, J. Physique statistique des surfaces aléatoires et combinatoire bijective des cartes planaires. PhD thesis, Université Paris VI, 2005.

[17] Bouttier, J., Di Francesco, P., Guitter, E., Planar maps as labeled mobiles. *Electronic J. Combinatorics* **11** (2004), #R69.

[18] Bouttier,J., Guitter,E. The three-point function of planar quadrangulations. *J. Stat. Mech. Theory Exp.* **(7)** P07020 (2008), 39p.

[19] Burago, D., Burago, Y., Ivanov, S. *A course in metric geometry* American Mathematic Society, (2001).

[20] Chassaing, P., Durhuus, B. Local limit of labeled trees and expected volume growth in a random quadrangulation. *Ann. Probab.* **34(3)** (2006) 879–917.

[21] Cori, R., Vauquelin, B. Planar maps are well labeled trees. *Canad. J. Math.* **33(5)** (1981), 1023–1042.

[22] Curien, N., Kortchemski, I. Percolation on random triangulations and stable looptrees. *Probab. Theory Related Fields* **163(1-2)** (2015), 303–337.

[23] Curien, N., Haas, B., Kortchemski, I., The CRT is the scaling limit of random dissections. *Random Struct. Alg.* **47(2)** (2015), 304–327.

[24] Curien, N., Le Gall, J.-F. Scaling limits for the peeling process on random maps. *Ann. Inst. Henri Poincaré* (B), to appear.

[25] Curien, N., Le Gall, J.-F. The hull process of the Brownian plane. *Probab. Th. Rel. Fields*, to appear.

[26] Dellacherie, C., Meyer, P.A., *Probabilités et Potentiel. Ch. I à IV.* Hermann, 1975.

[27] Derbez, E., Slade, G., The scaling limit of lattice trees in high dimensions. *Comm. Math. Phys.* **193** (1998), 69–104.

[28] Duplantier, B., Sheffield, S. Duality and the Knizhnik-Polyakov- Zamolodchikov relation in Liouville quantum gravity. *Phys. Rev. Lett.* (2009) **102(15)** :150603, 4.

[29] Duplantier, B., Sheffield, S. Liouville quantum gravity and KPZ *Inventiones mathematicae* **185**, Issue 2 (2011) 333–393 .

[30] Duquesne, T., Le Gall, J.-F., Probabilistic and fractal aspects of Lévy trees. *Probab. Theory Related Fields* **131** (2005), 553–603.

[31] Haas, B., Miermont, G., Scaling limits of Markov branching trees with applications to Galton-Watson and random unordered trees. *Ann. Probab.* **40** (2012), 2589–2666.

[32] Hambly, B., Lyons, T. Uniqueness for the signature of a path of bounded variation and the reduced path group. *Ann. of Math.* **(2) 171** (2010), 109–167.

[33] Hara, T., Slade, G., The scaling limit of the incipient infinite cluster in high-dimensional percolation. II. Integrated super-Brownian excursion. Probabilistic techniques in equilibrium and nonequilibrium statistical physics. *J. Math. Phys.* **41** (2000), 1244–1293.

[34] van der Hofstad, R., Slade, G., Convergence of critical oriented percolation to super-Brownian motion above 4+1 dimensions. *Ann. Inst. H. Poincaré Probab. Statist.* **39** (2003), 413–485.

[35] Itô, K. Poisson point processes attached to Markov processes. *Proceedings Sixth Berkeley Symposium on Mathematical Statistics and Probability*, vol. III, pp.225-239. University of California Press, Berkeley, 1972.

[36] Janson, S., Stefánsson, S.O., Scaling limits of random planar maps with a unique large face. *Ann. Probab.* **43(3)** (2015), 1045–1081.

[37] Krikun, M. Local structure of random quadrangulations. Preprint, available at `arXiv:0512304`.

[38] Lando, S.K., Zvonkin, A.K. Graphs on surfaces and their applications, Encyclopaedia of Mathematical Sciences, 141. Springer-Verlag, Berlin, 2004. With an appendix by Don B. Zagier, Low-Dimensional Topology, II.

[39] Lawler, G.F., Limic, V., *Random Walk : A Modern Introduction.* Cambridge Studies in Advanced Mathematics, 123. Cambridge University Press, Cambridge, 2010.

[40] Le Gall, J.-F., A class of path-valued Markov processes and its connections with superprocesses. *Probab. Th. Rel. Fields* **95** (1993), 25–46.

[41] Le Gall, J.-F., Itô's excursion theory and random trees. *Stoch. Process. Appl.* **120** (2010), 721–749.

[42] Le Gall, J.-F., Mouvement brownien, processus de branchement et superprocessus. Cours de DEA, Paris 6 (1994) available at `http://www.math.u-psud.fr/ jflegall/DEA94.pdf`.

[43] Le Gall, J.-F., Random trees and applications. *Probab. Surveys* **2** (2005), 245–311.

[44] Le Gall, J.-F., *Spatial Branching Processes, Random Snakes and Partial Differential Equations* Lectures in Mathematics, ETH Zürich. Birkhäuser, 1999.

[45] Le Gall, J.-F., The Brownian Cactus II. Upcrossings and local times of super-Brownian motion. *Probab. Th. Rel. Fields* **162** (2015), 199–231.

[46] Le Gall, J.-F., The Brownian snake and solutions of $\Delta u = u^2$ in a domain. *Probab. Th. Rel. Fields* **102** (1995), 393–432.

[47] Le Gall, J.-F., The topological structure of scaling limits of large planar maps. *Invent. Math.* **169** (2007), 621–670.

[48] Le Gall, J.-F., Uniqueness and universality of the Brownian map. *Ann. Probab.* **41** (2013), 2880–2960.

[49] Le Gall, J.-F., Miermont, G., Scaling limits of random trees and planar maps. Lecture notes from the 2010 Clay Mathematical Institute Summer School (Buzios). *Clay Mathematics Proceedings*, vol. **15** (2012), 155-211.

[50] Le Gall, J.-F., Paulin, F. Scaling limits of bipartite planarmaps are homeomorphic to the 2-sphere. *Geom. Funct. Anal.* **18(3)** (2008), 893–918.

[51] Le Gall, J.-F., Weill, M., Conditioned Brownian trees. *Annales Inst. H. Poincaré Probab. Stat.* **42** (2006), 455–489.

[52] Marckert, J.-F., Miermont, G., Invariance principles for random bipartite planar maps. *Ann. Probab.*, **35** (2007), 1642–1705.

[53] Marckert, J.-F., Mokkadem, A. Limit of normalized quadrangulations : the Brownian map. *Ann. Probab.* **34(6)** (2006), 2144–2202.

[54] Marckert, J.-F., Mokkadem, A., States spaces of the snake and of its tour – Convergence of the discrete snake. *J. Theoret. Probab.* **18** (2005), 615–625.

[55] Marckert, J.-F., Mokkadem, A., The depth first processes of Galton-Watson trees converge to the same Brownian excursion. *Ann. Probab.* **31** (2003), 1655–1678.

[56] Meyer, P.-A., Les inégalités de Burkholder en théorie des martingales. Séminaire de Probabilités. III (Univ. Strasbourg, 1967/68) pp. 163–174. Springer, Berlin, 1969.

[57] Miermont, G., Invariance principles for spatial multitype Galton-Watson trees. *Ann. Inst. Henri Poincaré* (B), **44** (2008), 1128–1161.

[58] Miermont, G. The Brownian map is the scaling limit of uniform random plane quadrangulations. *Acta Math.*, **210** (2013), 319–401.

[59] Miller, J., Sheffield, S., An axiomatic characterization of the Brownian map. Preprint, available at `arXiv:1506.03806`/

[60] Miller, J., Sheffield, S., Liouville quantum gravity and the Brownian map I : The QLE(8/3,0) metric. Preprint, available at `arXiv:1507.00719`

[61] Panagiotou, K., Stufler, B., Wellet, K. Scaling limits of random graphs from subcritical classes. *Ann. Probab.*, to appear.

[62] Pitman, J. *Combinatorial Stochastic Processes. Lecture Notes Math.* 1875. Springer, Berlin 2006.

[63] Revuz, D., Yor, M. *Continuous martingales and Brownian motion.* Springer, (2005).

[64] Rogers, L. C. G., Williams, D. *Diffusions, Markov processes and martingales, Volume 2.* Cambridge Mathematical Library, (2000)

[65] Schaeffer, G. Conjugaison d'arbres et cartes combinatoires aléatoires. PhD thesis. 1998.

[66] Stufler, B. The continuum random tree is the scaling limit of unlabelled unrooted trees. Preprint, available at `arXiv:1412.6333`.

[67] 't Hooft, G. A planar diagram theory for strong interactions. *Nuclear Physics B* **72** (1974), 461–473.

[68] Tutte, W.T. A census of Hamiltonian polygons. *Canad. J. Math.* **14** (1962), 402–417.

[69] Tutte, W.T. A census of planar triangulations. *Canad. J. Math.* **14** (1962), 21–38.

[70] Tutte, W.T. A census of slicings. *Canad. J. Math.* **14** (1962), 708–722.

[71] Tutte, W.T. A census of planar maps. *Canad. J. Math.* **15** (1963), 249–271.

Cartes aléatoires et serpent brownien

Mots-clés : cartes aléatoires, arbres aléatoires, théorie des excursions, serpent brownien.

Résumé

La première partie de cette thèse s'inscrit dans le domaine des cartes aléatoires, qui est un sujet à la frontière des probabilités, de la combinatoire et de la physique statistique. Nos travaux complètent une série de résultats de convergence de différents modèles de cartes aléatoires vers la carte brownienne, qui est un espace métrique compact aléatoire. Plus précisément, on montre que la limite d'échelle d'une carte de loi uniforme sur l'ensemble des cartes biparties enracinées à n arêtes, munie de la distance de graphe renormalisée par $(2n)^{-1/4}$, est, au sens de Gromov–Hausdorff, la carte brownienne. Pour prouver ce résultat, les arguments importants sont d'une part l'utilisation d'une bijection combinatoire entre cartes biparties et arbres multitypes, et d'autre part des théorèmes de convergence pour les arbres de Galton–Watson multitypes étiquetés.

Dans un deuxième temps, le but est de présenter une théorie des excursions pour le mouvement brownien indexé par l'arbre brownien. De manière analogue à la théorie d'Itô des excursions pour le mouvement brownien, chaque excursion correspond à une composante connexe du complémentaire des zéros du mouvement brownien indexé par l'arbre, et l'excursion est définie comme un processus indexé par un arbre continu. On explique comment mesurer la longueur de la frontière de ces excursions, de sorte que la famille de ces longueurs coïncide avec les sauts d'un processus de branchement à temps continu de mécanisme de branchement stable d'indice 3/2. De plus, conditionnellement aux longueurs des frontières, les excursions sont indépendantes et leur loi conditionnelle est déterminée à l'aide d'une mesure d'excursion explicite que l'on introduit et décrit. Dans ce travail, le serpent brownien apparaît comme un outil particulièrement important.

Abstract

The first part of this thesis concerns the area of random maps, which is a topic in between probability theory, combinatorics and statistical physics. Our work complements several results of convergence of various classes of random maps to the Brownian map, which is a random compact metric space. More precisely, we prove that the scaling limit of a map which is uniformly distributed over the class of rooted planar maps with n edges, equipped with the graph distance rescaled by $(2n)^{1/4}$, is, in the Gromov-Hausdorff sense, the Brownian map. To establish this result, the main arguments are the use of a combinatorial bijection between bipartite maps and multitype trees, together with convergence theorems for Galton-Watson multitype trees.

We then aim to develop an excursion theory for Brownian motion indexed by the Brownian tree. Analogous to the Itô excursion theory for Brownian motion, each excursion corresponds to a connected component of the complement of the zero set of the tree-indexed Brownian motion, and the excursion is defined as a process indexed by a continuous tree. We explain how to measure the length of the boundary of these excursions, in a way that the collection of these lengths coincides with the collection of jumps of a continuous-state branching process with a 3/2-stable branching mechanism. Moreover, conditionally on the boundary lengths, the excursions are independent and their conditional distribution is determined in terms of an excursion measure that we introduce and study. In this work, the Brownian snake appears as a particularly important tool.